XIV. *On the Inverse, or Inductive, Logical Problem.* By W. Stanley Jevons, M.A., F.R.S., Professor of Logic in The Owens College, Manchester.

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Logical deduction consists in ascertaining from a law or laws the combinations of qualities which may exist under those conditions. The natural law that all metals are conductors of electricity really means that in nature we find three classes of objects, namely:—(1) metals, conductors; (2) not metals, conductors; (3) not metals, not conductors. It comes to the same thing if we say that it excludes the existence of the class metals not conductors. But every scientific process has its inverse process. As addition is undone by subtraction, multiplication by division, involution by evolution, differentiation by integration, so logical induction is the inverse process of deduction. Given certain classes of objects, we endeavour by induction to pass back to the laws embodied in those classes; given combinations, we have to learn the laws obeyed by them. There does not exist, indeed, any distinct method of induction, except such as consists in inverting the processes of deduction, by noting and remembering the laws from which certain effects necessarily follow. The difficulties of induction are thus exactly analogous to those of integration, which can only be performed by trial, assisted by a full knowledge of the effects of differentiation and a certain happy knack of arranging the formulæ so as to bring them into connexion with some previous result of the direct process.

In two essays*, and in my paper "On the Mechanical Performance of Logical Inference"†, I have attempted to represent, with the utmost generality and simplicity, the processes of formal logic by which, from any proposition or series of propositions, we arrive at the combinations of terms possible under the condition of their truth. The inverse problem yet remains, I believe, to be considered. Given certain combinations, what are the propositions stating their conditions? I proceed to explain how this problem can be resolved in the case of two or three terms.

According to the laws of thought, two terms, say A and B, can be present or absent in four combinations, thus—

AB, A*b*, *a*B, *ab*.

A small italic letter indicates the absence or negation of the corresponding large one. The above combinations are unconditioned, except by the primary conditions of thought itself; but if we remove any one of the combinations, say A*b*, the meaning will be that A *which is not* B cannot exist. Thus, the three combinations AB, *a*B, *ab* being given, we pass back to the law all A's are B's, or, if A mean metal and B conductor, to a law such as "all metals are conductors."

We arrive at the utmost number of cases which can occur by omitting any one or more of the four combinations. The number of possible cases is therefore $2 \times 2 \times 2 \times 2$, or 16; and they are all shown in the following table, in which the sign 0 indicates the non-existence of the combination given at the left hand, and the mark 1 its presence.

* Pure Logic: London, 1864 (Stanford). The Substitution of Similars: London, 1869 (Macmillan).

† Philosophical Transactions, 1870, vol. clx. p. 497.

	1.	2.	3.	4.	5.	6.	7. *	8. *	9.	10. *	11.	12. *	13.	14. *	15. *	16. *
AB	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
A*b*	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
*a*B	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
ab	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Thus column 16 represents the case where all combinations are present, and the only conditions are the laws of thought. The example of metals and conductors of electricity would be represented in the 12th column; and every other mode in which two things or qualities might present themselves together or apart, is shown in one column or another. But more than half the cases may at once be rejected because they involve the entire absence of a term or its negative. Thus, in the 1st column, no combinations at all are represented as present. In column 2 there is only the negative combination *ab*. Now it is a logical principle that when any term or its negative entirely fails to appear, there must be some contradiction between the conditions of combination. Thus the two conditions A is B and A is not B, would result in altogether destroying the combinations containing A. We may therefore restrict our attention to those cases where at least two combinations are present, and they contain among them each of the letters A, B, *a*, *b*; these cases are represented in the columns marked with the sign *. Among these seven cases we find

> Four cases containing three combinations,
> Two cases containing two combinations,
> One case containing four combinations.

It has already been pointed out that a proposition of the form A is B, or, as it is more exactly represented in a symbolic notation explained in the essays referred to, A=AB, destroys one combination A*b*, and thus accounts for the 12th case. Let us consider in how many ways we can vary this form of proposition either by interchanging A and B or substituting for one or both of them its negative. We should arrive altogether at eight different expressions, which are thus stated:—

12.	8.	15.	14.
$A = AB$,	$A = Ab$,	$a = aB$,	$a = ab$,
$b = ab$,	$B = aB$,	$b = Ab$,	$B = AB$.

If we test the effect of each of these conditions by ascertaining the combinations which it negatives, it will be found that each proposition in the first line gives the same result as the one immediately below it. Each pair consists then of logical equivalents. And in fact the lower line contains what are called by logicians the contrapositives of those in the higher line. Thus the first pair mean

All A's are B's,
All not B's are not A's,

which have exactly the same logical force. The last pair mean

All not A's are not B's,
All B's are A's,

which are again exactly equivalent to each other. Although then there may be eight propositions of the form $A = AB$, only four of these have independent logical meanings. On trial we find that these four propositions give the combinations respectively shown in the 12th, 8th, 15th, and 14th columns, which cases are thus referred to their proper law.

If we now join these latter four propositions two and two, they will generally be found to contradict each other; thus all A's are B's contradicts all A's are not B's. There are only two pairs which give consistent results, namely:—

$$\left.\begin{array}{l} A = AB \\ a = ab \end{array}\right\}, \quad \text{and} \quad \left.\begin{array}{l} A = Ab \\ a = aB \end{array}\right\}.$$

The first of these pairs gives the combinations in the 10th column; but it may readily be shown that the two propositions all A's are B's, and all not A's are not B's, are but equivalent to the single proposition A=B, or all A's are all B's. In the same way the second pair of propositions gives the combinations of the 7th column, and are equivalent to the single proposition A=*b*, or all A's are all not B's. There remains but a single case, that in the 16th column; but as all the combinations are present, there can be no condition except the laws of thought.

We have now effected a complete solution of the inverse logical problem of two terms; we have found that two terms can manifest themselves only in seven series of combinations, and the corresponding laws are as below:—

Series of combinations.	Laws.	Equivalent laws.
12th	$A = AB$	$b = ab$
8th	$A = Ab$	$B = aB$
15th	$a = aB$	$b = Ab$
14th	$a = ab$	$B = AB$
10th	$A = B$	$a = b \begin{cases} A = AB \\ a = ab \end{cases}$
7th	$A = b$	$a = B \begin{cases} A = Ab \\ a = aB \end{cases}$
16th	No law	

We also learn from the above investigation that there is no possible logical relation between two terms which may not be expressed in a proposition either of the general type A=AB or of A=B. For the logical relation must manifest itself in some series of combinations of natural qualities; but every series of combinations which is possible, according to the very laws of thought, has been included in our investigation. Thus every such logical relation must either be expressed in one of the six laws, or must be equivalent to one of them. The general result of the problem then is that two terms admit only of six distinct logical relations, which again have only two essentially different typical forms, namely, A = AB and A = B. These laws express respectively partial and complete coincidence; the first is illustrated by the relation between *metals* and *elements*, the metals coinciding with a part of the elements; the latter, by the complete coincidence between substance possessing inertia, and substance possessing gravity, or between crystals of the cubical system and crystals not capable of doubly refracting light.

The Inverse Logical Problem involving Three Terms.

No sooner do we introduce into the problem a third term C, than the investigation assumes a far more complex character. There are now three terms, A, B, C, and their negatives, *a*, *b*, *c*, which may be combined, according to the Laws of Thought, in eight different combinations, namely,

ABC,	*a*BC,
AB*c*,	*a*B*c*,
A*b*C,	*ab*C,
A*bc*,	*abc*.

The effect of any logical conditions is to destroy one or more of these combinations. Now we may make selections from eight things in 2^8, or 256 ways; so that we have no less than 256 different cases to treat, and the complete solution is at least fifty times as troublesome as with two terms. Many series of combinations indeed are contradictory, as in the simpler problem, and may be passed over. The test of consistency is that each of the letters A, B, C, *a*, *b*, *c* shall appear somewhere in the series of combinations; but I have not been able to discover any mode of calculating the number of cases in which inconsistency would happen. The logical complexity of the problem is so great that the ordinary modes of calculating numbers of combinations in mathematical science fail to give any aid, and exhaustive examination of the combinations in detail is the only method applicable.

My mode of solving the problem was as follows:—Having written out the whole of the 256 series of combinations, I examined them separately, and struck out such as did not fulfil the test of consistency. I then chose some common form of proposition involving two or three terms, and varied it in every possible manner, both by the circular interchange of letters (A, B, C into B, C, A and then into C, A, B) and by the substitution for any one or more of the terms of the corresponding negative terms. For instance, the proposition AB = ABC can be first varied, by circular interchange, so as to give BC = BCA and then CA = CAB. Each of these three can then be thrown into eight varieties by negative change. Thus, AB = ABC gives aB = aBC, Ab = AbC, AB = ABc, ab = abC, and so on. Thus there may possibly exist no less than twenty-four varieties of the law having the general form AB = ABC, meaning that whatever has the properties of A and B has those also of C. It by no means follows that some of the varieties may not be equivalent to others; and trial shows, in fact, that AB = ABC is exactly the same in meaning as Ac = Acb or Bc = Bca. Thus the law in question has but eight varieties of distinct logical meaning. I now ascertain, by actual deductive reasoning, which of the 256 series of combinations result from each of these distinct laws, and mark them off as soon as found. I now proceed to some other form of law, for instance A = ABC, meaning that whatever has the qualities of A has those also of B and C. I find that it admits of twenty-four variations, all of which are found to be logically distinct; the combinations being worked out, I am able to mark off twenty-four more of the list of 256 series. I proceed in this way to work out the results of every form of law which I can find or invent. If in the course of this work I obtain any series of combinations which had been previously marked off, I learn at once that the law is logically equivalent to some law previously treated. It may be safely inferred that every variety of the apparently new law will coincide in meaning with some variety of the former expression of the same law; I have sufficiently verified this assumption in some cases, and have never found it lead to error. Thus, just as $AB = ABC$ is equivalent to $Ac = Abc$, so we find that $ab = abC$ is equivalent to $ac = acB$.

Among the laws treated were the two $A = AB$ and $A = B$, which involve only two terms, because it may of course happen that among three things two only are in special logical relation, and the third independent; and the series of combinations representing such cases of relation are sure to occur in the complete enumeration. All single propositions which I could invent having been treated, pairs of propositions were next investigated. Thus we have the relations "all A's are B's, and all B's are C's," of which the old logical syllogism is the development. We may also have "all A's are all B's, and all B's are C's," or even "all A's are all B's, and all B's are all C's." All such premises admit of variations, greater or less in number, the logical distinctness of which can only be determined by trial in detail. Disjunctive propositions, either singly or in pairs, were also treated, but were often found to be equivalent to other propositions of a simpler form; thus $A = B \cdot|\cdot C$ (the sign $\cdot|\cdot$ standing for the disjunctive conjunction *or* in an unexclusive sense) is exactly the same in meaning as $a = bc$.

This mode of exhaustive trial bears some analogy to that ancient mathematical process called the sieve of Eratosthenes. Having taken a long series of the natural numbers, Eratosthenes is said to have calculated out in succession all the multiples of every number, and to have marked them off, so that at last the prime numbers alone remained, and the factors of every number were exhaustively discovered. My problem of 256 series of combinations is the logical analogue, the chief points of difference being that there is a limit to the number of cases, and that prime numbers have no analogue in logic, since every series of combinations corresponds to some law or group of conditions. But the analogy is perfect in the point that they are both inverse processes. There is no mode of ascertaining that a number is prime but by showing that it is not the product of any assignable factors. So there is no mode of ascertaining what laws are embodied in any series of combinations but trying exhaustively the laws which would give them. Just as the results of Eratosthenes's method have been worked out to a great extent and registered in tables for the convenience of other mathematicians, I have endeavoured to work out the inverse logical problem to the utmost extent which is at present practicable or useful.

I have thus found that there are altogether fifteen conditions or series of conditions which may govern the combinations of three terms, forming the premises of fifteen essentially different kinds of arguments. The following table contains a statement of these conditions, together with the number of combinations which are contradicted or destroyed by each, and the number of logically distinct variations of which the law is capable. There might be also added, as a sixteenth case, that case in which no special logical condition exists, so that all the eight combinations remain.

Reference Number.	Propositions expressing the general type of the logical conditions.	Number of distinct logical variations.	Number of combinations contradicted by each.
I.	A=B	6	4
II.	A=AB	12	2
III.	A=B, B=C	4	6
IV.	A=B, B=BC	24	5
V.	A=AB, B=BC	24	4
VI.	A=BC	24	4
VII.	A=ABC	24	3
VIII.	AB=ABC	8	1
IX.	A=AB, *a*B=*a*B*c*	24	3
X.	A=ABC, *ab*=*ab*C	8	4
XI.	AB=ABC, *ab*=*abc*	4	2
XII.	A=ABC ·|· A*bc*	12	2
XIII.	A=BC ·|· A*bc*	8	3
XIV.	A=BC ·|· *bc*	2	4
XV.	A=ABC, *a*=B*c* ·|· *b*C	8	5
		192	

There are 63 series of combinations derived from self-contradictory premises, which, with the above 192 series and the one case where there are no conditions or laws at all, make up the whole conceivable number of 256 series.

We learn from this table, for instance, that two propositions of the form A=AB, B=BC, which are such as constitute the premises of the old syllogism Barbara, negative or render impossible four of the eight combinations in which three terms may be united, and that these propositions are capable of taking twenty-four variations by transpositions of the terms or the introduction of negatives. This table then presents the results of a complete analysis of all the possible logical relations arising in the case of three terms, and the old syllogism forms but one out of fifteen typical forms. Generally speaking, every form can be converted into apparently different propositions; thus the fourth type A=B, B=BC may appear in the form A=ABC, *a*=*ab*, or again in the form of three propositions A=AB, B=BC, *a*B=*a*B*c*; but all these sets of premises yield identically the same series of combinations, and are, therefore, of exactly equivalent logical meaning. The fifth type, or Barbara, can also be thrown into the equivalent form $A=ABC$, $aB=aBC$. In other cases I have obtained the very same logical conditions in four modes of statement. As regards mere appearance and mode of statement, the number of possible premises would be almost unlimited.

The most remarkable of all the types of logical conditions is the fourteenth, namely $A=BC \cdot|\cdot bc$. It is that which expresses the division of a genus into two doubly marked species, and might be illustrated by the example—Component of the physical Universe = Matter, gravitating, ·|· Not matter (ether), not gravitating.

It is capable of only two distinct logical variations, namely $A=BC \cdot|\cdot bc$ and $A=Bc \cdot|\cdot bC$. By transposition, or negative change of the letters, we can indeed obtain six different expressions of each of these propositions; but when their meanings are analyzed by working out the combinations, they are found to be logically equivalent to one or the other of the above two. Thus the proposition $A=BC \cdot|\cdot bc$ can be written in any of the following five other modes:—

$$a=bC \cdot|\cdot Bc, \qquad B=CA \cdot|\cdot ca, \qquad b=cA \cdot|\cdot Ca,$$
$$C=AB \cdot|\cdot ab, \qquad c=aB \cdot|\cdot Ab.$$

I do not think it needful at present to publish the complete table of 193 series of combinations and the premises corresponding to each. Such a table enables us by mere inspection to learn the laws obeyed by any set of combinations of three things, and is to logic what a table of factors and prime numbers is to the theory of numbers, or a table of integrals to the higher mathematics. The table already given above (p. 128) would enable a person with but little labour to discover the law of any combinations. If there be seven combinations (one contradicted) the law must be of the eighth type, and the proper variety will be apparent; if there be six combinations (two contradicted), either the second, eleventh, or twelfth type applies, and a certain number of trials will disclose the proper type and variety; if there be but two combinations, the law must be of the third type; and so on.

The above investigations are complete as regards the possible logical relations of two or three terms. But when we attempt to apply the same kind of method to the relations of four or more terms, the labour becomes impossibly great. Four terms give sixteen combinations compatible with the laws of thought, and the number of possible selections of combinations is no less than 2^{16}, or 65,536. The following table shows the extraordinary manner in which the number of possible logical relations increases with the number of terms involved.

Number of terms.	Number of possible combinations.	Number of possible selections of combinations corresponding to consistent or inconsistent logical relations.
2	4	16
3	8	256
4	16	65,536
5	32	4,294,967,296
6	64	18,446,744,073,709,551,616

Some years of continuous labour would be required to ascertain the precise number of types of laws which may govern the combinations of only four things; and only a small part of such laws would be exemplified or capable of practical application in science. The purely logical inverse problem whereby we pass from combinations to their laws is solved in the preceding pages as far as it is likely to be for a long time to come; and it is almost impossible that it should ever be carried more than a single step further.