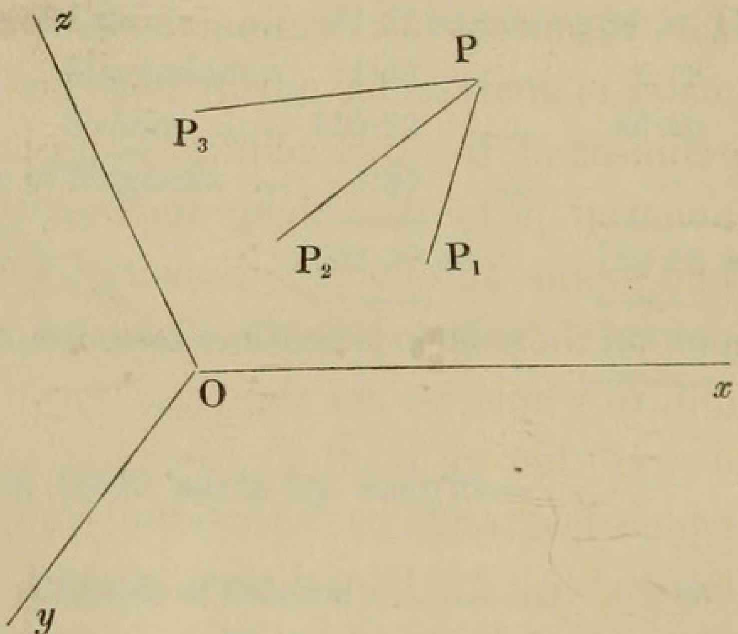
## XVII.—*New Investigation of* LAPLACE'S *Theorem, in the Theory of Attractions.* POISSON'S *Remarks on this Theorem.* By ROBERT RAWSON, Esq.

(Read December 7th, 1847.)




LET each of any number of material points, $P_1$; $P_2$; $P_3$; &c. &c. attract a material point P with a force inversely as the squares of the distances $PP_1$; $PP_2$; &c. &c. The resultant of these forces in amount and direction, when the boundary of the attracting points is any specified figure or surface, has been the great object of investigation by mathematicians since the theory of attractions was first developed by Newton, in his great work the Principia.

No subject of inquiry, since Newton's time, has been attended with greater difficulties, or received the attention of more profound and enlightened minds, than the theory of attractions. In pursuing the investigation, we are conducted to all the resources and refinements with which the domains of analysis have been enriched by the great geniuses of the last century, who have left upon immortal pages the deep impress of their great and noble intellects. Some of the most illustrious of these are Newton, James and John Bernoulli, to whom we are indebted for the useful and well known method of integration by parts, which Lagrange has successfully applied in order to establish his fine theory of the Calculus of Variations; Leibnitz, Euler, D'Alembert, Laplace, Lagrange, Poisson, Simpson, Maclaurin,—and Ivory, who has the merit of being the first in this country who studied the works of the continental writers. All of these have enriched science with valuable artifices, modes of investigation, and results which they have obtained by means of successfully developing the artifices and new views which they have created.

To investigate the resultant of the forces $P_1$; $P_2$ &c. &c. which act upon the point P, it will be necessary in the first place, to refer all the points in the system to three planes $x$ O $y$; $x$ O $z$; $y$ O $z$ at right angles to each other and arbitrarily fixed. We then resolve each force in the direction of these three fixed co-ordinate planes and sum the effects of the attractions in these directions, then these sums will enable us to calculate the resultant in magnitude and direction by the well known formula $R = \sqrt{A^2 + B^2 + C^2}$ and $\frac{A}{R} = \text{Cos.}\alpha$; $\frac{B}{R} = \text{Cos.}\beta$; $\frac{C}{R} = \text{Cos.}\gamma$; where R = The resultant and A, B, C the sum of the forces in the direction of the co-ordinate axes $x$, $y$, $z$ respectively; and $\alpha$, $\beta$, $\gamma$ the angles which R makes with the co-ordinate planes. (See Poisson's Traite Mécanique, page 55.)

This mode of investigation was first laid down and pursued by the celebrated Maclaurin in his Treatise on Fluxions. See Mécanique Analytique, page 227, where Lagrange has enumerated the principal steps in the science of mechanics, and by whom they were made.

I may here state that besides the relations above given we have

$$\text{Cos.}^2\alpha + \text{Cos.}^2\beta + \text{Cos.}^2\gamma = 1$$

$$R = A\,\text{Cos.}\,\alpha + B\,\text{Cos.}\,\beta + C\,\text{Cos.}\,\gamma$$

Let $x_1$, $y_1$, $z_1$ be the rectangular co-ordinates of the point $P_1$, origin at O.

| | | | | |
|---|---|---|---|---|
| $x_2$, $y_2$, $z_2$ | do. | do. | $P_2$ | do. |
| &c. | | &c. | &c. | |

And $a$, $b$, $c$ be the rectangular co-ordinates of the point P

Put $P_1 P = w_1$ and $P_2 P = w_2$ &c. &c.

Let $\overset{x}{\Sigma}$ ; $\overset{y}{\Sigma}$ ; $\overset{z}{\Sigma}$ refer to all the material points situated in the lines parallel to axis $x$, $y$, $z$ respectively, and each passing through the point P.

And let $\overset{xy}{\Sigma}$, $\overset{xz}{\Sigma}$, $\overset{yz}{\Sigma}$ refer to points placed in planes parallel to the co-ordinate planes $xy$, $xz$, $yz$ respectively passing through the point P, excepting those points situated in lines passing through P and parallel respectively to $x$, $y$, & $z$.

$\Sigma$ refers to all the other points of the system.

Then we have $\frac{\rho}{w_1^2}$ = the attraction of $P_1$ on P in the direction of $P_1P$, where $\rho$ is a constant and equal to $P_1$'s amount of attraction at a unit of distance from P.

The cosines of the angles which $P_1P$ makes with the axis of $x$, $y$, $z$ respectively will be $\frac{a-x_1}{w_1}$ $\frac{b-y_1}{w_1}$; $\frac{c-z_1}{w_1}$ (See Poisson's Mécanique, p. 171.)

Hence, if we resolve $\frac{\rho}{w_1^2}$ parallel to the co-ordinate axis by multiplying it by the cosines of the angles which it makes with the co-ordinate planes we shall have—

$$\left.\begin{array}{ll} \Sigma\cdot\frac{\rho(a-x_1)}{w_1^3} = \text{force parallel to the axis of } & x \\ \Sigma\cdot\frac{\rho(b-y_1)}{w_1^3} = \quad \text{do.} \qquad \text{do.} & y \\ \Sigma\cdot\frac{\rho(c-z_1)}{w_1^3} = \quad \text{do.} \qquad \text{do.} & z \end{array}\right\} (1)$$

These forces are derived entirely from the material points which are not placed in the planes parallel to the co-ordinate planes, and passing through P.

$$\left.\begin{array}{l}\overset{x}{\Sigma}\cdot\dfrac{\rho}{w_1^2} = \text{the force of all the material points} \\ \qquad\text{parallel to } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \ x \\ \overset{y}{\Sigma}\cdot\dfrac{\rho}{w_1^2} = \qquad \text{do.} \qquad\qquad \text{do.} \qquad\quad y \\ \overset{z}{\Sigma}\cdot\dfrac{\rho}{w_1^2} = \qquad \text{do.} \qquad\qquad \text{do.} \qquad\quad z\end{array}\right\}(2)$$

These are the forces derived from the material points placed in the lines parallel to the co-ordinate axes $x$, $y$, and $z$ respectively.

$$\left.\begin{array}{l}\overset{x\,y}{\Sigma}\cdot\dfrac{\rho(a-x_1)}{w_1^3} = \text{the force parallel to } x\ldots\ldots\ldots\ldots \\ \overset{x\,y}{\Sigma}\cdot\dfrac{\rho(b-y_1)}{w_1^3} = \qquad \text{do.} \qquad \text{do.} \qquad y\ldots\ldots\ldots\ldots\end{array}\right\}(3)$$

The force parallel to $z$ will be nothing from $\overset{x\,y}{\Sigma}\cdot$

$$\left.\begin{array}{l}\overset{x\,z}{\Sigma}\cdot\dfrac{\rho(a-x_1)}{w_1^3} = \text{force parallel to } x \ \ldots\ldots\ldots\ldots\ldots \\ \overset{x\,z}{\Sigma}\cdot\dfrac{\rho(c-z_1)}{w_1^3} = \qquad \text{do.} \qquad z\ \ldots\ldots\ldots\ldots\ldots\end{array}\right\}(4)$$

The force parallel to $y$ will be nothing from $\overset{x\,y}{\Sigma}\cdot$

$$\left.\begin{array}{l}\overset{y\,z}{\Sigma}\cdot\dfrac{\rho(b-y_1)}{w_1^{\,3}} = \text{force parallel to } y \text{ ...............} \\ \overset{y\,z}{\Sigma}\cdot\dfrac{\rho(c-z_1)}{w_1^{\,3}} = \quad \text{do.} \qquad z \text{ ...............}\end{array}\right\}(5)$$

The force from $\overset{y\,z}{\Sigma}$ being nothing parallel to $x$.

Collecting the forces parallel to axes $x$, $y$, $z$ respectively we shall have—

$$A = \Sigma\cdot\frac{\rho(a-x_1)}{w_1^{\,3}} + \overset{x\,y}{\Sigma}\cdot\frac{\rho(a-x_1)}{w_1^{\,3}} + \overset{x\,z}{\Sigma}\cdot\frac{\rho(a-x_1)}{w_1^{\,3}} + \overset{x}{\Sigma}\cdot\frac{\rho}{w_1^{\,2}} \ldots(6)$$

$$B = \Sigma\cdot\frac{\rho(b-y_1)}{w_1^{\,3}} + \overset{x\,y}{\Sigma}\cdot\frac{\rho(b-y_1)}{w_1^{\,3}} + \overset{y\,z}{\Sigma}\cdot\frac{\rho(b-y_1)}{w_1^{\,3}} + \overset{y}{\Sigma}\cdot\frac{\rho}{w_1^{\,2}} \ldots(7)$$

$$C = \Sigma\cdot\frac{\rho(c-z_1)}{w_1^{\,3}} + \overset{x\,z}{\Sigma}\cdot\frac{\rho(c-z_1)}{w_1^{\,3}} + \overset{y\,z}{\Sigma}\cdot\frac{\rho(c-z_1)}{w_1^{\,3}} + \overset{z}{\Sigma}\cdot\frac{\rho}{w_1^{\,2}} \ldots(8)$$

These are the forces which act parallel to the co-ordinate axes where the values of $\Sigma\cdot w_1$ &c. &c., in terms of the co-ordinates of the point attracted and the point attracting are as follows.

$$\left.\begin{array}{l}
\Sigma\cdot w_1 = \Sigma\cdot\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)\right\}^{\frac{1}{2}}\,;\quad \overset{x\,y}{\Sigma}\cdot w_1 = \overset{x\,y}{\Sigma}\cdot\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{1}{2}} \\[2ex]
\overset{x\,z}{\Sigma}\cdot w_1 = \overset{x\,z}{\Sigma}\cdot\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{1}{2}}\,;\quad \overset{y\,z}{\Sigma}\cdot w_1 = \overset{y\,z}{\Sigma}\cdot\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{1}{2}} \\[2ex]
\overset{x}{\Sigma}\cdot w_1 = \overset{x}{\Sigma}\cdot(a-x_1)\,;\quad \overset{y}{\Sigma}\cdot w_1 = \overset{y}{\Sigma}\cdot(b-y_1)\ \ \&\ \ \overset{z}{\Sigma}\cdot w_1 = \overset{z}{\Sigma}\cdot(c-z_1)
\end{array}\right\}\ (9)$$

Let us now substitute the values of $\Sigma\cdot w_1$ &c. &c. as given in the system of equations (9), in the equations (6), (7), and (8), and we shall have—

$$\left.\begin{aligned}
\frac{\mathrm{A}}{\rho} &= \Sigma\cdot\frac{(a-x_1)}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}} + \overset{x\,y}{\Sigma}\cdot\frac{(a-x_1)}{\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{3}{2}}} + \overset{x\,z}{\Sigma}\cdot\frac{(a-x_1)}{\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}} + \overset{x}{\Sigma}\cdot\frac{1}{(a-x_1)^2} \\
\frac{\mathrm{B}}{\rho} &= \Sigma\cdot\frac{(b-y_1)}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}} + \overset{x\,y}{\Sigma}\cdot\frac{(b-y_1)}{\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{3}{2}}} + \overset{y\,z}{\Sigma}\cdot\frac{(b-y_1)}{\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}} + \overset{y}{\Sigma}\cdot\frac{1}{(b-y_1)^2} \\
\frac{\mathrm{C}}{\rho} &= \Sigma\cdot\frac{(c-z_1)}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}} + \overset{x\,z}{\Sigma}\cdot\frac{(c-z_1)}{\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}} + \overset{y\,z}{\Sigma}\cdot\frac{(c-z_1)}{\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}} + \overset{z}{\Sigma}\cdot\frac{1}{(c-z_1)^2}
\end{aligned}\right\} \quad (10)$$

We must remember that the sum of the forces of the material particles which are on one side of the attracted point, with respect to the three planes passing through it and parallel to the co-ordinate planes, will be positive, and the sum of the forces of the material particles on the other side will be negative, this will necessarily give rise, generally, to as many negative terms in the equations (10) as positive, in consequence of the material particle placed at P being acted upon by the difference between the forces on one side of it and those on the other.

If we follow the steps of Laplace, and put

$$\frac{\mathrm{V}}{\rho} = \Sigma\cdot\frac{1}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{1}{2}}} + \overset{x\,y}{\Sigma}\cdot\frac{1}{\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{1}{2}}} + \overset{x\,z}{\Sigma}\cdot\frac{1}{\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{1}{2}}}$$

$$+\overset{y\,z}{\Sigma}\cdot\frac{1}{\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{1}{2}}} + \overset{x}{\Sigma}\cdot\frac{1}{(a-x_1)} + \overset{y}{\Sigma}\cdot\frac{1}{(b-y_1)} + \overset{z}{\Sigma}\cdot\frac{1}{(c-z_1)} \ldots\ldots\ldots\ldots \quad (11)$$

Differentiate this equation with respect to $a$, $b$, &c. respectively, and we shall have—

$$\left.\begin{aligned}
-\frac{dV}{\rho da}&=\Sigma\cdot\frac{(a-x_1)}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}}+\overset{x\,y}{\Sigma}\cdot\frac{(a-x_1)}{\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{3}{2}}}+\overset{x\,z}{\Sigma}\cdot\frac{(a-x_1)}{\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}}+\overset{x}{\Sigma}\cdot\frac{1}{(a-x_1)^2}\\
-\frac{dV}{\rho db}&=\Sigma\cdot\frac{(b-y_1)}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}}+\overset{x\,y}{\Sigma}\cdot\frac{(b-y_1)}{\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{3}{2}}}+\overset{y\,z}{\Sigma}\cdot\frac{(b-y_1)}{\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}}+\overset{y}{\Sigma}\cdot\frac{1}{(b-y_1)^2}\\
-\frac{dV}{\rho dc}&=\Sigma\cdot\frac{(c-z_1)}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}}+\overset{x\,z}{\Sigma}\cdot\frac{(c-z_1)}{\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}}+\overset{y\,z}{\Sigma}\cdot\frac{(c-z_1)}{\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}}+\overset{z}{\Sigma}\cdot\frac{1}{(c-z_1)^2}
\end{aligned}\right\}\quad(12)$$

Hence, by referring to the system of equations (10), we shall have the following equations obtained by Laplace by means of the above beautiful artifice of partial differentiation.

$$\frac{d\,V}{d\,a} = -\,A\,; \quad \frac{d\,V}{d\,b} = -\,B \;\&\; \frac{d\,V}{d\,c} = -\,C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

If we differentiate equations (12) again with respect to $a$, $b$, $c$ respectively, we shall have—

$$-\frac{d^2V}{\rho\,d\,a^2} = \Sigma\cdot\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{-\frac{3}{2}} - 3\,(a-x_1)^2\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{-\frac{5}{2}}$$

$$+\overset{x\;y}{\Sigma}\cdot\left\{(a-x_1)^2+(b-y_1)^2\right\}^{-\frac{3}{2}} - 3\,(a-x_1)^2\left\{(a-x_1)^2+(b-y_1)^2\right\}^{-\frac{5}{2}}$$

$$+\overset{x\;z}{\Sigma}\cdot\left\{(a-x_1)^2+(c-z_1)^2\right\}^{-\frac{3}{2}} - 3\,(a-x_1)^2\left\{(a-x_1)^2+(c-z_1)^2\right\}^{-\frac{5}{2}} - 2\,\overset{x}{\Sigma}\cdot\frac{1}{(a-x_1)^3}$$

$$\therefore -\frac{d^2V}{\rho\,d\,a^2} = \Sigma\cdot\frac{(b-y_1)^2+(c-z_1)^2-2\,(a-x_1)^2}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{5}{2}}} + \overset{x\;y}{\Sigma}\cdot\frac{(b-y_1)^2-2\,(a-x_1)^2}{\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{5}{2}}}$$

$$+\overset{x\;z}{\Sigma}\cdot\frac{(c-z_1)^2-2\,(a-x_1)^2}{\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{5}{2}}} - 2\overset{x}{\Sigma}\cdot\frac{1}{(a-x_1)^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

Similarly we shall obtain—

$$-\frac{d^2V}{\rho\, d b^2} = \Sigma \cdot \frac{(a-x_1)^2+(c-z_1)^2-2\,(b-y_1)^2}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{5}{2}}} + \overset{x\ y}{\Sigma} \cdot \frac{(a-x_1)^2-2\,(b-y_1)^2}{\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{5}{2}}}$$

$$+ \overset{y\ z}{\Sigma} \cdot \frac{(c-z_1)^2-2\,(b-y_1)^2}{\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{5}{2}}} - 2\overset{y}{\Sigma} \cdot \frac{1}{(b-y_1)^3} \cdots\cdots\cdots \quad (15)$$

$$-\frac{d^2V}{\rho\, d c^2} = \Sigma \cdot \frac{(a-x_1)^2+(b-y_1)^2-2\,(c-z_1)^2}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{5}{2}}} + \overset{x\ z}{\Sigma} \cdot \frac{(a-x_1)^2-2\,(c-z_1)^2}{\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{5}{2}}}$$

$$+ \overset{y\ z}{\Sigma} \cdot \frac{(b-y_1)^2-2\,(c-z_1)^2}{\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{5}{2}}} - 2\overset{z}{\Sigma} \cdot \frac{1}{(c-z_1)^3} \cdots\cdots\cdots \quad (16)$$

By adding equations (14), (15) and (16) together, we shall have—

$$\frac{d^2V}{\rho d a^2} + \frac{d^2V}{\rho d b^2} + \frac{d^2V}{\rho d c^2} = \overset{x\,y}{\Sigma}\cdot \frac{1}{\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{3}{2}}} + \overset{x\,z}{\Sigma}\cdot \frac{1}{\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}}$$

$$+\overset{y\,z}{\Sigma}\cdot \frac{1}{\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}} + 2\overset{x}{\Sigma}\cdot\frac{1}{(a-x_1)^3} + 2\overset{y}{\Sigma}\cdot\frac{1}{(b-z_1)^3} + 2\overset{z}{\Sigma}\cdot\frac{1}{(c-z_1)^3} \cdots\cdots \quad (17)$$

If we suppose $\overset{X\,Y}{\Sigma}\cdot$, $\overset{X\,Z}{\Sigma}\cdot$ and $\overset{Y\,Z}{\Sigma}\cdot$ to refer to all the points in the planes X Y &c. &c. parallel to the co-ordinate planes, and passing through the material point P; we shall have—

$$\frac{V}{\rho} = \Sigma\cdot \frac{1}{\left\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\right\}^{\frac{1}{2}}} + \overset{X\,Y}{\Sigma}\cdot \frac{1}{\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{1}{2}}} + \overset{X\,Z}{\Sigma}\cdot \frac{1}{\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{1}{2}}}$$

$$+\overset{Y\,Z}{\Sigma}\cdot \frac{1}{\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{1}{2}}} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (18)$$

And equation (17) will become—

$$\frac{d^2V}{\rho da^2}+\frac{d^2V}{\rho db^2}+\frac{d^2V}{\rho dc^2}=\overset{X\ Y}{\Sigma}\cdot\frac{1}{\left\{(a-x_1)^2+(b-y_1)^2\right\}^{\frac{3}{2}}}$$

$$+\overset{X\ Z}{\Sigma}\cdot\frac{1}{\left\{(a-x_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}}+\overset{Y\ Z}{\Sigma}\cdot\frac{1}{\left\{(b-y_1)^2+(c-z_1)^2\right\}^{\frac{3}{2}}}\quad(19)$$

$\frac{V}{\rho}$ = sum of every point in the mass of the attracting body multiplied by the reciprocal of its distance from P.

If the point P which is attracted, be exterior to all the attracting points, and such that none of the attracting bodies are in the planes passing through P, and parallel to the co-ordinate planes. Then we shall have from (19).

$$\frac{d^2V}{d\,a^2}+\frac{d^2V}{d\,b^2}+\frac{d^2V}{d\,a^2}=0\ldots\ldots\ldots\ldots\ldots\ldots\ (20)$$

This equation was first given by the justly celebrated mathematical philosopher Laplace, (See Pratt's Mechanical Philosophy, page 155,) and Poisson was the first to show that the theorem was not true when the attracted point P was surrounded by the attracting particles, and his reasoning on this question, in order to make the equation continuous, I must confess, has always appeared to me to be difficult to comprehend. The mode Poisson has adopted, is to divide the attracting body into two parts, one of which is a sphere surrounding the attracted point, and the other of course is the remaining shell, for every material point of which shell he supposes Laplace's equation to be true, and he then calculates the effects of the sphere upon the point P. This mode of reasoning has conducted Poisson to the equation

$$\frac{d^2V}{d\,a^2} + \frac{d^2V}{d\,b^2} + \frac{d^2V}{d\,c^2} = -4\,\pi\rho^1 \dots\dots\dots\dots\dots \quad (21)$$

which he states to be true when the attracted particle is a part of the attracting mass. I am however, as before stated, unable to see the force of Poisson's reasoning on this question, and certainly the conclusion to which he arrives is different from the result which I have obtained in equation (19) by means of a very different Analysis. The condition which must be complied with, in order that Laplace's equation may be true is, from equation (19), that the points attracting must not be situated in any one of the three planes, passing through the point P attracted, parallel to the co-ordinate planes.

And if the attracted point P be a part of the attracting mass, then equation (19) will give the relation between the three partial differential co-efficients of V with respect to the co-ordinates of the attracted point. I state these conclusions with great diffidence and respect for the high authority of Poisson, whose results on this subject appear to me to be different from those which the foregoing investigation has enabled me to obtain; and I cannot refrain from thinking, in consequence of being unable to alter the above researches, that Poisson's correction of Laplace's equation when the attracted point is a part of the attracting mass is not strictly right. The error which I conceive Poisson has made is, in stating that the shell into which he divides the attracting body will satisfy Laplace's equation.

Thus suppose $V = V^1 + U$, where $V^1$ refers to the shell and $U$ to the sphere which surrounds the attracted point.

$$\therefore \frac{d^2V}{da^2} = \frac{d^2V^1}{da^2} + \frac{d^2U}{da^2}$$

$$\& \frac{d^2V}{db^2} = \frac{d^2V^1}{db^2} + \frac{d^2U}{db^2}$$

$$\& \frac{d^2V}{dc^2} = \frac{d^2V^1}{dc^2} + \frac{d^2U}{dc^2}$$

$$\therefore \frac{d^2V}{da^2} + \frac{d^2V}{db^2} + \frac{d^2U}{dc^2} = \frac{d^2V^1}{da^2} + \frac{d^2V^1}{db^2} + \frac{d^2V^1}{dc^2} + \frac{d^2U}{da^2}$$
$$+ \frac{d^2U}{db^2} + \frac{d^2U}{dc^2}$$

Now Poisson supposes that

$$\frac{d^2V^1}{da^2} + \frac{d^2V^1}{da^2} + \frac{d^2V^1}{dc^2} = 0 \ldots\ldots\ldots\ldots\ldots \quad (22)$$

$$\therefore \frac{d^2V}{da^2} + \frac{d^2V}{db^2} + \frac{d^2V}{dc^2} = \frac{d^2U}{da^2} + \frac{d^2U}{db^2} + \frac{d^2U}{dc^2} \ldots \quad (23)$$

The equation (22) is not, I believe, correct; it is established by Poisson from the commutability of the independent operations of $d$, & $\int f(x_1 a)\, dx$ where the integral sign $\int$ refers to $dx$, and the differential sign $d$ refers to $a$; this principle only obtains when the limits of integration with respect to $x$ are independent of $a$.

Thus, $\dfrac{dV}{da} = \dfrac{dV^1}{da} + \dfrac{dU}{da}$

where $-\frac{dV^1}{da}$ is the force arising from the shell and $-\frac{dU}{da}$ the force arising from the sphere—

$\frac{V}{\rho}=\iiint\frac{dx\,.\,dy\,.\,dz}{\left\{(a-x)^2+(b-y)^2+(c-z)^2\right\}^{\frac{1}{2}}}$ where the rectangular co-ordinates $x$ $y$ & $z$ refers to any point in the attracting mass—(See Poisson Traité Mécanique, Nos. 96 and 107)—and the limits of integration depends upon the equation of the surface of the attracting mass. Differentiate the above equation with respect to $a$ then we shall have—

$$\frac{-dV}{\rho da}=\iiint\frac{(a-x)\,dx\,dy\,dz}{\left\{(a-x)^2+(b-y)^2+(c-z)^2\right\}^{\frac{3}{2}}}$$

consequently,

$$\frac{A}{\rho}=\iiint\frac{(a-x)\,dx\,dy\,dz}{\left\{(a-x)^2+(b-y)^2+(c-z)^2\right\}^{\frac{3}{2}}}\;\ldots\ldots\;(24)$$

Similarly we have

$$\frac{B}{\rho}=\iiint\frac{(b-y)\,dx\,dy\,dz}{\left\{(a-x)^2+(b-y)^2+(c-z)^2\right\}^{\frac{3}{2}}}\;\ldots\ldots\;(25)$$

$$\& \ \frac{C}{\rho} = \iiint \frac{(c-z)\,dx\,dy\,dz}{\left\{(a-x)^2+(b-y)^2+(c-z)^2\right\}^{\frac{3}{2}}} \ldots\ldots (26)$$

See Poisson's Mécanique, No. 96. The limits of these integrations in the case of the shell $V^1$ are functions of $a$, $b$, $c$, the co-ordinates of the attracted point. Hence the differential of A with respect to $a$ cannot be performed, in the case of $V^1$, before the integration with respect to $x$, $y$, $z$. These reasons have induced me to believe that Poisson's correction of Laplace's theorem is wrong.

And for the limits of integration in the equations (24), (25), and (26) to be independent of the co-ordinates of P, which must be the case if Laplace's equation is true, we must have the following conditions—

$$\text{If, } f(x, y, z) = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

be the equation to the surface of the attracting mass

$$\left.\begin{aligned} \therefore\ & f(a, y, z) = 0 \\ \&\ & f(x, b, z) = 0 \\ \&\ & f(x, y, c) = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

will be the equations to three plane sections of the attracting mass passing through P, the attracted point, and parallel to the co-ordinate planes $yz$, $xz$, $xy$ respectively. If the first of equation (28) for instance, gives impossible values for $y$, when $z$ takes any value whatever &c. with the second and third, then Laplace's equation is true, if to the contrary it is not necessarily true.