# Art. XIX.—*Methods of Election.*

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If there be several candidates for an office of any kind, and the appointment rests in the hands of several persons, an election is held to decide who is to receive the appointment. The object of such an election is to select, if possible, some candidate who shall, in the opinion of a majority of the electors, be most fit for the post. Accordingly, the fundamental condition which must be attended to in choosing a method of election is that the method adopted must not be capable of bringing about a result which is contrary to the wishes of the majority. There are several methods in use, and none of them satisfy this condition. The object of this paper is to prove this statement, and to suggest a method of election which satisfies the above condition.

Let us suppose, then, that several persons have to select one out of three or more candidates for an office. The methods which are in use, or have been put forward at various times, may be divided into three classes.

The first class includes those methods in which the result of an election is arrived at by means of a single scrutiny.

The second class includes those in which the electors have to vote more than once.

The third class includes those in which more than one scrutiny may be necessary, but in which the electors have only to vote once.

In describing these methods, the number of candidates will in some cases be supposed to be any whatever, but in other cases it will be assumed, for the sake of simplicity, that there are only three candidates. The case in which there are only three candidates is the simplest, and it is of frequent occurrence. I propose, therefore, to examine, for the case of three candidates, the results of the methods which have been proposed, and to show that they are erroneous in this case. This will be sufficient for my purpose, for it will be easily seen that the methods will be still more liable to error if the number of candidates be greater than three. I shall then discuss at some length the proposed method in the case of three candidates, and afterwards consider more briefly the case of any number of candidates.

## Methods of the First Class.

In the first class three methods may be placed, viz., the single vote method, the double vote method, and the method of Borda. In these methods the electors have only to vote once, and the result is arrived at by means of a single scrutiny.

## The Single Vote Method.

This is the simplest of all methods, and is the one adopted for Parliamentary elections in all English-speaking communities in the case in which there is only one vacancy to be filled. As is well known, each elector has one vote, which he gives to some one candidate, and the candidate who obtains the greatest number of votes is elected. This method is used for any number of candidates; but in general the larger the number of candidates the more unsatisfactory is the result.

In this method, unless some candidate obtains an absolute majority of the votes polled, the result may be contrary to the wishes of the majority. For, suppose that there are twelve electors and three candidates, A, B, C, who receive respectively five, four, and three votes. Then A, having the largest number of votes, is elected. This result, however, may be quite wrong; for it is quite possible that the four electors who vote for B may prefer C to A, and the three electors who vote for C may prefer B to A. If this were the case, and the question

That A is to be preferred to B

were put to the whole body of electors, it would be negatived by a majority of two, and the question

That A is to be preferred to C

would also be negatived by a majority of two. Thus the single vote method places at the head of the poll a candidate who is declared by a majority of the electors to be inferior to each of the other candidates. In fact, if A and B were the only candidates B would win; or if A and

C were the only candidates C would win; thus B and C can each beat A, and yet neither of them wins. A wins simply because he is opposed by two men, each better than himself.

Thus the single vote method does not satisfy the fundamental condition. It appears also not only that the best man may not be elected, but also that we are not even sure of getting in the second best man. It is clear that if any candidate obtain an absolute majority of the votes polled this error cannot occur. All we can say, then, about the single vote method is that if any candidate obtain an absolute majority the method is correct, but if no one obtains such a majority the result may be quite erroneous.

These results are well known, and consequently in elections under this plan great efforts are generally made to reduce the number of candidates as much as possible before the polling day, in order to avoid the return of a candidate who is acceptable to a small section only of the electors. This reduction can, in practice, be made only by a small number of the electors, so that the choice of a candidate is taken out of the hands of the electors themselves, who are merely permitted to say which of two or more selected candidates is least objectionable to them.

### The Double Vote Method.

In this method each elector votes for two candidates, and the candidate who obtains the largest number of votes is elected. This method is erroneous, for it may lead to the rejection of a candidate who has an absolute majority of votes in his favour, as against all comers. For suppose that there are twelve electors, and that the votes polled are, for A, nine; for B, eight; for C, seven, then A is elected. Now, in order to show that this result may be erroneous it is merely necessary to observe that it is possible that each of the seven electors who voted for C may consider C better than A and B; that is to say, an absolute majority of the electors may consider C to be the best man, and yet the mode of election is such that not only does C fail to win, but in addition he is at the bottom of the poll. This is an important result; we shall see presently the effect it has on other methods of election.

In the case in which there are only three candidates this method is, in fact, equivalent to requiring each elector to vote against one candidate, and then electing the candidate who has the smallest number of votes recorded against him.

## Borda's Method.

This method was proposed by Borda in 1770, but the first published description of it is in the volume for 1781 of the *Memoirs of the Royal Academy of Sciences.* For some remarks on the method see Todhunter's *History of Probability*, p. 433, where the method is described. In the case of three candidates, it is as follows. Each elector has three votes, two of which must be given to one candidate, and the third vote to another candidate. The candidate who obtains the greatest number of votes is elected.

In order to show that this method may lead to an erroneous result, suppose that there are twelve electors, of whom five prefer A to B and B to C, whilst two prefer A to C and C to B, and five prefer B to C and C to A. Then the votes polled will be, for A, fourteen; for B, fifteen; for C, seven. Thus B is elected. It is clear, however, that this result is wrong, because seven out of the whole twelve electors prefer A to B and C, so that, in fact, A has an absolute majority of the electors in his favour. Hence, then, Borda's method does not satisfy the fundamental condition, for it may lead to the rejection of a candidate who has an absolute majority of the electors in his favour.

It may be observed that the result of the poll on Borda's method may be obtained, in the case of three candidates, by adding together the corresponding results in the polls on the methods already described.

If there be $n$ candidates, each elector is required to arrange them in order of merit; then for each highest place $n—1$ votes are counted; for each second place, $n—2$ votes, and so on; $n—r$ votes being counted for each $r^{\text{th}}$ place, and no votes for the last place. The candidate who obtains the greatest number of votes is elected.

Borda does not give any satisfactory reason for adopting the method. Nevertheless he had great faith in it, and made use of it to test the accuracy of the ordinary or single vote method, and arrived at the extraordinary conclusion that in any case in which the number of candidates is equal to or exceeds the number of electors, the result cannot be depended upon unless the electors are perfectly unanimous. This in itself is sufficient to show that Borda's method must be capable of bringing about a result which is contrary to the wishes of the majority.

There is, however, another objection which is of great importance. Borda's method holds out great inducements to the electors to vote otherwise than according to their real views. For if an elector strongly desires the return of a particular candidate, he not only gives his two votes to that candidate, but he also takes care to give his remaining vote to the least formidable of the other candidates. The effect of this is to give a great advantage to second-rate candidates. Thus not only does Borda's method fail to interpret the true wishes of the electors, supposing that they vote honestly, but it holds out great inducements to them to vote otherwise than according to their real views.

Laplace discussed the question of the best mode of electing one out of several candidates, and by an analytical investigation was led to Borda's method.* He states distinctly that this method is the one indicated by the theory of probabilities. He then proceeds to point out the objection just stated, and expresses the opinion that the method would, without doubt, be the best if each elector would write the names of the candidates in what he thinks the order of merit. We have seen, however, that this is far from being the case.

## Methods of the Second Class.

The simplest method of the second class is the French method of double elections. In this method each elector has one vote, as in the single vote method, already described. If, however, no candidate obtain an absolute majority of the votes polled, a second election is held. For this second election only the two candidates who obtained the largest number of votes at the first election can be candidates. The result is that the successful candidate is returned by an absolute majority of those who vote at the second election, so that it would appear, at first sight, that the successful candidate represents the views of a majority of the electors. We must not lose sight, however, of two facts, first, that all the electors who vote at the first election may not vote at the second election ; second, that those who do so vote merely have to choose between the two remaining candidates, and that, consequently, they may not be repre-

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* *Journal de l'Ecole Polytechnique*, cahiers vii. and viii., pp. 169, 170; *Théorie Analytique des Probabilités*, pp. 101, 299; Todhunter's *History of Probability*, pp. 547, 548.

sented in any sense by the candidate they vote for; they may merely be in the position of having a choice of evils.

This plan has frequently been proposed for adoption in England, and quite recently it has been proposed by more than one speaker in the Legislative Assembly of Victoria. The method is indeed a great improvement on the present system of single voting, and if the election be merely a party contest, and neither side runs more than two candidates, the result cannot be wrong. But if these conditions be not satisfied, the method may easily lead to an erroneous result. The method may be used whatever be the number of candidates; but it is sufficient to show that it is erroneous in the case of three candidates only. This is at once done by a further consideration of the example already given in discussing the single vote method. For in that example C is at the bottom of the poll, and, according to the present system, he is rejected, and a second election is held to decide between A and B, because no one has an absolute majority at the first election. The result of the second election is, for A, five votes; for B, seven votes; so that B wins. In order to show that this result may be erroneous it is only necessary to suppose that the five electors who voted for A prefer C to B. For then, if the question

That C is to be preferred to B

was put to the whole body of electors, it would be carried by a majority of four. Now, we have already seen that the question

That C is to be preferred to A

would be carried by a majority of two. Hence, then, this method leads to the rejection of a candidate who is declared by a majority of the electors to be superior to each of the other candidates. This method, then, clearly violates the condition that the result must not be contrary to the wishes of the majority.

We may consider this example from a slightly different point of view. In discussing it under the single vote method, the important result arrived at was that A was inferior to each of the other candidates, and, therefore, ought to be at the bottom of the poll, instead of being at the top, as he was, in consequence of his being opposed by two good men, B and C. Thus, instead of excluding C, as in the French method, A is the one who ought to be excluded. Having arrived at the result that A is to be excluded, the whole of the electors have now a right to decide between B and C. On putting this question to the issue, we find that C is preferred by the electors.

We see, then, that the French method may lead to error through throwing out the best man at the first election. And this is the only way in which it can err; for if there be a best man, and he survive the ordeal of the first election, he must win at the second, seeing that he is, in the opinion of the electors, better than each of his competitors.

Comparing the French method with the single vote method, we see that in the case of three candidates the worst candidate may be returned by the single vote method, but that it would be impossible for such a result to be brought about by the French method. By that method we are at least sure of getting the second best man, if we fail to get the best.

There is, however, a grave practical objection to this method. It is that a second polling may be necessary. This is of great importance; for in the case where the number of electors is large, as in a political election, great expense has to be incurred, not only by the authorities in providing the necessary machinery, but also by the electors themselves in coming to the poll again. Besides this, the excitement of the election is kept up much longer than it would be if the whole matter could be settled by a single polling. There can, I think, be little doubt that this objection has been one of the chief obstacles with which the advocates of this method have had to contend. Accordingly, we find that the single vote method is employed, as a rule, in those cases in which there are some hundreds of electors, and it would be inconvenient to hold a second election. On the other hand, when the number of electors is small, so that they can all meet together, and remain till a second or third election has been held, the number of candidates is generally reduced to two by means of a preliminary ballot or ballots. This very fact shows that the defects of the single vote method are recognised, because in those cases in which it is considered to be practicable to do so a preliminary election is held, so as to try to avoid the glaring defect of the single vote method—that is, to avoid returning a candidate who is acceptable to a small section only of the electors. It is a mistake, however, to suppose that it is not practicable to hold one or more preliminary elections when the number of electors is large. It is generally thought that in order to do so a fresh set of voting papers must be used for the second election, and that this second election cannot be held till the result of the first is known, so that the electors have the expense and trouble of going to the poll a second time. This, at all events, appears to be the practice in France, Germany, and Italy. This, however, is not necessary; for, by a very simple expedient, any number of preliminary elections, on any plan whatever, may be held by means of a single set of voting papers, and without troubling the electors to vote more than once. The expedient is to require each elector to indicate his order of preference amongst all the candidates. Once get this information from the electors, and we can tell how any elector will vote on any question that may be put as to the merits of the candidates. It is here assumed that an elector will not change his opinion during the course of the election. This expedient of making each elector indicate his order of preference amongst all the candidates is necessary in order to carry out Borda's method, which has been described above; indeed, it was suggested by Borda himself. But Borda does not appear to have noticed that it might be made use of for a series of elections without requiring the electors to vote again; this appears to have been first pointed out by Condorcet. The idea of a preferential or comparative voting paper is one of the fundamental ones in Hare's system of proportional representation. We are not concerned with this subject here, as the only question under consideration is that of filling a single vacancy. It is, however, worthy of notice that the preferential voting paper which is such an important feature in Hare's system, is of such old origin, and that it was suggested by Condorcet as a means of filling several vacancies, which is the very question considered by Hare. The method of Condorcet, however, is quite different to that of Hare.

If the expedient here described were adopted, the French system would be free from the practical objection which has been indicated. It would still, however, be open to the objection that the result of the election might be contrary to the views of the electors. Notwithstanding this, the method would be a good practical one for elections on a large scale; it would be very suitable for party contests, and if neither side ran too many candidates, the result could not be wrong. The method, however, would be altogether unsuitable if there were three distinct parties to the contest. Under any circumstances, however, the method would be very little more complicated than the present system of single voting, and it would give much better results. If, however, it be considered desirable to reform the present electoral system so far as to introduce this French system of double elections, it would be as well to at once adopt the method of Ware, described below. This is the same, in the case of three candidates, as the French method, but in other cases it is a trifle longer. No difference whatever would be required in the method of voting, but only a little more labour on the part of the returning officer. The results of this method would be much more trustworthy than those of the French method.

## Other Methods of the Second Class.

Before passing on to the methods of the third class, it may be stated that each of the methods described under that heading may be conducted on the system of the second class. In order to do so, instead of using a preferential voting paper, as in the methods of the third class, we must suppose a fresh appeal made to the electors after each scrutiny. This, of course, would make the methods needlessly complex, and, in the case of a large number of electors, totally impracticable. This, however, is not the only objection to the methods of the second class. For if the electors be allowed to vote again after the result of one of the preliminary elections is known, information is given which may induce an elector to transfer his allegiance from a candidate he has been supporting to another candidate whom he finds has more chance of success. A method which permits, and which even encourages, electors to change their views in the middle of the contest cannot be considered perfect. This objection does not apply to those cases in which there are only three candidates, or to any case in which all but two candidates are rejected at the first preliminary election, as in the French system.

There is another objection, however, which applies to all cases alike; it is that, at the first preliminary election, an astute elector may vote, not according to his real views, but may, taking advantage of the fact that there is to be a second election, vote for some inferior candidate in order to get rid, at the first election, of a formidable competitor of the candidate he wishes to win. If this practice be adopted by a few of the supporters of each of the more formidable competitors, the result will frequently be the return of an inferior man.

On account of these objections, I consider it unnecessary to enter into any further details as to the methods of the second class.

## Methods of the Third Class.

In the methods of the third class each elector makes out a list of all the candidates in his order of preference, or, what comes to the same thing, indicates his order of preference by writing the successive numbers 1, 2, 3, &c., opposite the names of the candidates on a list which is supplied to him. Thus one voting only is required on the part of the electors. These preferential or comparative lists are then used in a series of scrutinies; and the methods of the third class differ from one another only in the way in which these scrutinies are conducted. Three different methods, which may be called Ware's method, the Venetian method, and Condorcet's practical method, have been proposed for use, and these will now be described.

## Ware's Method.

This method is called Ware's method because it appears to have been first proposed for actual use by W. R. Ware, of Harvard University.* The method was, however, mentioned by Condorcet,† but only to be condemned. This method is a perfectly feasible and practicable one for elections on any scale, and it has recently been adopted by the Senate of the University of Melbourne. It is a simple and obvious extension of the French system, and it is obtained from that system by two modifications, viz.:—

(1.) The introduction of the preferential or comparative method of voting, so as to dispense with any second voting on the part of the electors.

(2.) The elimination of the candidates one by one, throwing out at each scrutiny the candidate who has fewest votes, instead of rejecting at once all but the two highest.

In the case in which there are three candidates only, the second modification is not necessary. It will, perhaps, be convenient to give a more formal description of this method. The mode of voting for all methods of the third class has already been described; it remains, therefore, to describe

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* See *Hare on Representation*, p. 353.

† *Œuvres*, 1804, vol. xiii., p. 243.

the mode of conducting the scrutinies in Ware's method.

At each scrutiny each elector has one vote, which is given to the candidate, if any, who stands highest in the elector's order of preference.

The votes for each candidate are then counted, and if any candidate has an absolute majority of the votes counted he is elected.

But if no candidate has such an absolute majority, the candidate who has fewest votes is excluded, and a new scrutiny is proceeded with, just as if the name of such excluded candidate did not appear on any voting paper.

Successive scrutinies are then held until some candidate obtains on a scrutiny an absolute majority of the votes counted at that scrutiny. The candidate who obtains such absolute majority is elected.

It is obvious that this absolute majority must be arrived at sooner or later.

It is clear, also, that if on any scrutiny any candidate obtain a number of votes which is greater than the sum of all the votes obtained by those candidates who each obtain less than that candidate, then all the candidates having such less number of votes may be at once excluded.

Ware's method has been shown to be erroneous for the case of three candidates in the remarks on the French method, of which it is in that case a particular form. It is easy to see that if there be more than three candidates the defects of this method will be still more serious.

The objection to this method, concisely stated, is that it may lead to the rejection of a candidate who is considered by a majority of the electors to be better than each of the other candidates. At the same time, the method is a great improvement on the single vote method; and the precise advantage is that whereas the single vote method might place at the head of the poll a candidate who is considered by a majority of the electors to be worse than each of the other candidates, it would be impossible for such a candidate to be elected by Ware's method.

To illustrate fully the difference between the two methods and the defects of each, suppose that there are several candidates, A, B, C, D, . . P, Q, R, and that in the opinion of the electors each candidate is better than each of the candidates who follow him in the above list, so that A is clearly the best, B the second best, and so on, R being the worst. Then on the single vote method R may win; on Ware's method A, B, C, D, . . P, may be excluded one after another on the successive scrutinies, and at the final scrutiny the contest will be between Q and R, and Q, of course, wins, since we have supposed him better than R in the opinion of the electors. Thus the single vote method may return the worst of all the candidates; and although Ware's method cannot return the worst, it may return the next worst.

A great point in favour of Ware's method is that it is quite impossible for an astute elector to gain any advantage for a favourite candidate by placing a formidable competitor at the bottom of the list. On account of its simplicity, Ware's method is extremely suitable for political elections. In cases of party contests, the strongest party is sure to win, no matter how many candidates are brought forward. The successful candidate, however, will not always be the one most acceptable to his own party.

## The Venetian Method.

For the sake of simplicity, I describe this method for the case of three candidates only. Two scrutinies are held; at the first scrutiny each elector has two votes, which are given to the two candidates, one to each, who stand highest in the elector's order of preference. The candidate who has fewest votes is then rejected, and a final scrutiny is held between the two remaining candidates. At the final scrutiny each elector has one vote, which is given to that one of the remaining candidates who stands highest in the elector's order of preference. The candidate who obtains most votes at the final scrutiny is elected.

This method is very faulty; it may lead to the rejection of a candidate who has an absolute majority of the electors in his favour. For we have seen, in discussing the double vote method, that such a candidate may be rejected at the first scrutiny. In fact, unless the candidate who has fewest votes at the first scrutiny has less than N votes, where 2N is the number of electors, we cannot be sure the result is correct. For, for anything we can tell, the candidate who is rejected at the first scrutiny may be, in the opinion of an absolute majority of the electors, the best man for the post. If, however, the candidate who has fewest votes on the first scrutiny has less than N votes, then the method will certainly give a correct result. For, since there are only three candidates, to require an elector to vote for two candidates comes to exactly the same thing as to ask him to vote against one candidate. Now, if with the two votes any candidate get less than N votes, it is clear that there are more than N votes against him, for each candidate must be marked first, or second, or third on each paper. Thus, in the opinion of an absolute majority, the candidate is worse than each of the other candidates, and, therefore, ought not to be elected. Unless, therefore, the lowest candidate has less than N votes, this method violates the fundamental condition.

I do not know that the method has ever been used in the form here described; but in the still more objectionable form of the second class, which differs from the one just described only by dispensing with the preferential voting paper, and allowing the electors to vote again after the result of the first scrutiny is known, it is exceedingly common, and is frequently used by committees. An instance which was fully reported in the Melbourne papers occurred some time ago in the selection of a candidate to stand on the constitutional side at the last election for Boroondara. It is fair, however, to say that the result of the method appears to have been correct in that case; but that was due to accident, and not to the method itself.

If there be more than three candidates the method is very complicated, and the defects are more serious. It seems, however, hardly worth while going into any details in these cases.

## Condorcet's Practical Method.

This method was proposed in 1793 by Condorcet, and appears to have been used for some time at Geneva. It is described at pp. 36—41 of vol. xv. of Condorcet's collected works (edition of 1804), and may be used in the case of any number of candidates for any number of vacancies. We are at present concerned only with the case of a single vacancy; and for the sake of simplicity I describe Condorcet's method for the case in which there are only three candidates.

Two scrutinies may be necessary in order to ascertain the result of the election in this method. At the first scrutiny one vote is counted for each first place assigned to a candidate, and if any candidate obtains an absolute majority of the votes counted he is elected. But if no one obtain such an absolute majority a second scrutiny is held. At the second scrutiny one vote is counted for each first place, and one vote for each second place, exactly as in the first scrutiny on the Venetian method, and the candidate who obtains most votes is elected. At first sight we might suppose that this method could not lead to error. Comparing it with the Venetian method, described above, we see that Condorcet supplies a remedy for the obvious defect of the Venetian method—that is to say, the rejection of a candidate who has an absolute majority is now impossible. A little examination, however, will show, as seems to have been pointed out by Lhuilier,* that the method is not free from error. For, let us suppose that there are sixteen electors, of whom five put A first and B second, five put C first and B second, two put A first and C second, two put B first and A second, and two put C first and A second. Then the result of the first scrutiny will be, for A, B, C, seven, two, seven votes respectively. Thus, no one having an absolute majority, a second scrutiny is necessary. The result of the second scrutiny will be—for A, B, C, eleven, twelve, and nine votes respectively. Thus B, having the largest number of votes, is elected. This result, however, is not in accordance with the views of the majority of the electors. For the proposition, "B is better than A," would be negatived by a majority of two votes, and the proposition, "B is better than C," would also be negatived by a majority of two votes, so that in the opinion of the electors B is worse than A and also worse than C, and, therefore, ought not to be elected.

Summing up the results we have arrived at, we see that each of the methods which have been described may result in the return of a candidate who is considered by a majority of the electors to be inferior to each of the other candidates. Some of the methods—viz., the double vote method, the method of Borda, and the Venetian method—may even result in the rejection of a candidate who has an absolute majority of votes in his favour as against all comers. It would, however, be quite impossible for such a result to occur on the single vote method, or the methods of Ware and Condorcet.

## Method Proposed.

Having pointed out the defects of the methods in common use, it now remains to describe the method proposed for adoption, and to show that it is free from these defects. It

* See Montucla's *Histoire des Mathématiques*, vol. iii., p. 421.

consists merely in combining the principle of successive scrutinies with the method of Borda, and at the same time making use of the preferential voting paper, so that the proposed method belongs to the third class. I propose, first, to describe and discuss the method for the case of three candidates, and then to pass on to the general case in which there may be any number of candidates.

Let us suppose, then, that there are three candidates, A, B, C. Each elector writes on his voting paper the names of two candidates in order of preference, it being clearly unnecessary to write down a third name. If we prefer it, the three names may be printed on the voting paper, and the elector may be required to indicate his order of preference by writing the figure 1 opposite the name of the candidate of his first choice, and the figure 2 opposite the name of the candidate of his second choice, it being clearly unnecessary to mark the third name. In order to ascertain the result of the election two scrutinies may be necessary.

At the first scrutiny two votes are counted for each first place and one vote for each second place, as in the method of Borda. Then if the two candidates who have the smallest number of votes have each not more than one-third of the whole number of votes, the candidate who has most votes is elected, as in Borda's method. But if one only of the candidates has not more than one-third of the votes polled (and some candidate must have less), then that candidate is rejected, and a second scrutiny is held to decide between the two remaining candidates. At the second scrutiny each elector has one vote, which is given to that one of the remaining candidates who stands highest in the elector's order of preference. The candidate who obtains most votes at the second scrutiny is elected.

The method may be more briefly described as follows:—Proceed exactly as in Borda's method, but instead of electing the highest candidate, reject all who have not more than the average number of votes polled. If two be thus rejected, the election is finished; but if one only be rejected, hold a final election between the two remaining candidates on the usual plan.

In order to show that the proposed method is free from the defects above described, it is necessary and it is sufficient to show that if the electors consider any one candidate, A, say, superior to each of the others, B and C, then A cannot be rejected at the first scrutiny. For if A be not rejected at the first scrutiny he cannot fail to win at the second scrutiny. Let therefore the whole number of electors be 2N, and let the number who prefer B to C be $N + a$, and consequently the number who prefer C to B be $N—a$; similarly, let the number who prefer C to A be $N + b$, and therefore the number who prefer A to C be $N — b$, and let the number who prefer A to B be $N + c$, and therefore the number who prefer B to A be $N — c$. Then it is easy to see that the numbers of votes polled by A, B, C at the first scrutiny will be

$$2N - b + c,\ 2N - c + a,\ 2N - a + b$$

respectively. For if the compound symbol A B be used to denote the number of electors who put A first and B second, and similarly for other cases, it is clear that A's score at the first scrutiny will be

$$2AB + 2AC + BA + CA.$$

Now this expression can be written in the form

$$(AB + AC + CA) + (AC + AB + BA),$$

and it is clear that the three terms in the first pair of brackets represent precisely the number of electors who prefer A to B, which number has already been denoted by $N + c$. In the same way the remaining three terms represent the number of electors who prefer A to C, which number has been denoted by $N — b$. Hence the score of A on the first scrutiny is $2N — b + c$. In exactly the same way it may be shown that the scores of B, C are $2N — c + a$ and $2N — a + b$ respectively. The sum of these three numbers is 6N, as it ought to be. Thus 2N is the mean or average of these three numbers, and consequently the highest of the three candidates must have more than 2N votes, and the lowest must have less than 2N votes. Now, let us suppose that a majority of the electors prefer A to B, and likewise that a majority prefer A to C; then $c$ must be positive, and $b$ must be negative. Hence the score of A, which has been shown to be $2N — b + c$, is necessarily greater than 2N, for it exceeds 2N by the sum of the two positive quantities $— b$ and $c$. Thus A has more than 2N votes, that is, more than one-third, or the average of the votes polled. He cannot, therefore, be rejected at the first scrutiny, so that B or C or both must be rejected at the first scrutiny. If either of the two, B and C, be not rejected, A must win at the second scrutiny, for there is a majority for A against B, and also against C. Hence, then, it has been demonstrated that if the opinions of the electors are such that there is a majority in favour of A as against B, and likewise a majority in favour of A as against C, the method of election which is proposed will certainly bring about the correct result; whereas it has been shown by the consideration of particular examples that the methods in ordinary use may easily bring about an erroneous result under these circumstances. Thus the proposed method cannot bring about a result which is contrary to the wishes of the majority, so that the proposed method satisfies the fundamental condition.

The method which is proposed has, I think, strong claims. It is not at all difficult to carry out. The result will, as often as not, be decided on the first scrutiny. We simply require each elector to put down the names of two of the three candidates in order of preference. Then for each first name two votes are counted, and for each second name one vote is counted. The number of votes for each candidate is then found. The third part of the sum total may be called the average; then all candidates who are not above the average are at once rejected. The lowest candidate must, of course, be below the average. The second is just as likely to be below as above the average. If he is below, the election is settled; but if he is above the average, a second scrutiny is necessary to decide between him and the highest candidate.

## Cases of Inconsistency.

We have now to consider what is the result of the proposed method in those cases in which there is not a majority for one candidate against each of the others. The methods which have been described have been shown to be erroneous by examining cases in which either one candidate has an absolute majority of the electors in his favour, or a candidate A is inferior to B and also to C, or a candidate A is superior to B and also to C. Now it is not necessary that any of these cases should occur. If a single person has to place three candidates in order of preference he can do so, and it would be quite impossible for any rational person to arrive at the conclusions

B is superior to C ... ... ... (1)
C is superior to A ... ... ... (2)
A is superior to B ... ... ... (3)

When, however, we have to deal with a body of men, this result may easily occur, and no one of the candidates can be elected without contradicting some one of the propositions stated above. If this result does occur, then, no matter what result any method of election may give, it cannot be demonstrated to be erroneous. We have examined several methods, and all but the one now proposed have been shown to lead to erroneous results in certain cases. It may fairly be urged, then, that that method which cannot be shown to be erroneous in any case has a greater claim to our consideration than any of the other methods which can be shown to erroneous. On this ground alone I think the method proposed ought to be adopted for all cases.

We can, however, give other reasons in favour of the method proposed. We have seen that it gives effect to the views of the majority in all cases except that in which the three results (1), (2), (3) are arrived at. In this case there is no real majority, and we cannot arrive at any result without abandoning some one of the three propositions (1), (2), (3). It seems most reasonable that that one should be abandoned which is affirmed by the smallest majority. Now, if this be conceded, it may be shown that the proposed method will give the correct result in all cases. For it is easily seen that the majorities in favour of the three propositions (1), (2), (3) are respectively $2a$, $2b$, $2c$. Hence, then, in the case under consideration, $a$, $b$, $c$, must be all positive. Let us suppose that $a$ is the smallest of the three. Then we abandon the proposition (1), and consequently C ought to be elected. Now let us see what the proposed method leads to in this case. B's score at the first scrutiny is $2N - c + a$, and this is necessarily less than 2N, because $c$ is greater than $a$, and each is positive. Again, C's score is $2N - a + b$, and this is necessarily greater than 2N, because $b$ is greater than $a$, and each is positive. Thus B is below the average, and C is above the average. Therefore, at the first scrutiny B goes out and C remains in. If A goes out also, C wins at the first scrutiny. But if A does not go out, C will beat A at the second scrutiny. Thus C wins in either case, and, therefore, the proposed method leads to the result which is obtained by abandoning that one of the propositions (1), (2), (3) which is affirmed by the smallest majority. We have already seen that in the case in which the numbers $a$, $b$, $c$ are not all of the same sign, the proposed method leads to the correct result. Hence, then, if it be admitted that when we arrive at the three inconsistent propositions (1), (2), (3) we are to abandon the one which is affirmed by the smallest majority, it follows that the proposed method will give the correct result in all cases.

We have, then, arrived at two results. First, that if the electors affirm any two of the propositions (1), (2), (3) and affirm the contrary of the remaining one, and so affirm three consistent propositions, then the result of the method of election which is here proposed, will be that which is the logical consequence of these propositions, whilst the methods in ordinary use may easily give a different result. Second, that if the electors affirm the three propositions (1), (2), (3) which are inconsistent, then the result of the method proposed is that which is the logical consequence of abandoning that one of the three propositions which is affirmed by the smallest majority.

## Another way of Applying Proposed Method.

The method may be stated in another form, which may sometimes be more convenient. For each first place count one vote; then, if any candidate has an absolute majority, elect him. But if not, count in addition one vote for each second place; then, if the lowest candidate has not got half as many votes as there are electors, reject him, and proceed to a final scrutiny between the remaining two. But, if not, take the aggregate for each candidate of the results of the two counts; then reject all who have less than one-third of the votes now counted, and, if necessary, proceed to a final scrutiny.

This process will give the same final result as the method already described. This is readily seen as follows:—1st. If any one has an absolute majority on the first places, the election is settled at the first scrutiny, and the result is manifestly correct, and therefore the same as that of the proposed method. 2nd. If no one has an absolute majority on the first places, but some one has on first and second places less than half as many votes as there are electors, it is manifest that more than half the electors consider that candidate worse than each of the others, so that he ought to be rejected, and hence the result of the final scrutiny will be correct, and therefore in accordance with that of the proposed method. 3rd. If neither of the above events happen, we take the aggregate. Now (as has already been remarked) the result of taking the aggregate is to give us exactly the same state of the poll as in the first scrutiny of the proposed method. Thus the second way of applying the method will give the same final result as the proposed method. This second way is very convenient, for if there be an absolute majority for or against any candidate, it is made obvious at the first or second count, and the election is settled with as little counting as possible. The two counts are conducted on well known plans, and if the circumstances are such that either of these necessarily gives a correct result, that result is adopted. But if it is not obvious that a correct result can be arrived at, then we take the mean, or what comes to the same thing, the aggregate of the two counts. This might appear to be a rule of thumb, and on that account may perhaps commend itself to some persons. This is not the case, however; and it is remarkable that that which might suggest itself as a suitable compromise in the matter should turn out to be a rigorously exact method of getting at the result in all cases. The view of the proposed method which has just been given shows exactly what modifications require to be made in Condorcet's practical method in order to make it accurate.

## Laplace's Objection.

It may be said that the proposed method is open to the objection raised by Laplace to the method of Borda. To this I think it a sufficient answer to say, that if we have a method which will truly interpret the wishes of the electors, as expressed by their voting papers, we need not trouble ourselves whether they vote honestly or not; that is their own concern. If we provide a method which will bring out a correct result for honest electors we need not try to go further, and endeavour to construct a method which will force dishonest electors to vote honestly. Nevertheless, it may be pointed out that Laplace's objection is not of so much force in this case as in the case of Borda's method. For if an elector vote otherwise than according to his real views it will be at the risk of having his vote at the final scrutiny counted against the candidate whom he considers most fit for the office to be filled. This risk would be sufficient to deter most electors from voting otherwise than according to their real opinions. If, in spite of this risk, an elector persists in voting otherwise than according to his real views we must take him at his word. To illustrate this objection, let us suppose that B and C are two formidable candidates, and that A is in reality inferior to each of them, but that the voting is as follows, BA = 5, CA = 4, AB = 1, AC = 1; so that B's supporters, in their anxiety to defeat C, put A second, and C's supporters, in their anxiety to defeat B, put A second. The result at the first scrutiny is A 13 votes, B 11 votes, C 9 votes. Thus C is rejected and A wins in the final scrutiny. A wins because the whole of C's supporters put him second. Had one of C's supporters voted according to his real views, and put B second, the result would have been different.

If the preferential mode of voting were not employed, this objection would be of great force; for then the supporters of each candidate would put his most formidable opponent at the bottom of their list at the first scrutiny, knowing that they would have at the second scrutiny an opportunity of reviewing their vote.

### A Modification of Proposed Method.

It may be mentioned that there is another, but in general a more tedious, method of getting at a result, which cannot be shown to be erroneous in any case. This method has been adopted by the Trinity College Dialectic Society. It is as follows:—In the method proposed above, instead of rejecting all the candidates who are not above the average, reject the lowest only. It is obvious from what has been said above that this cannot lead to error. But a second scrutiny will always be required, whereas in the proposed method one scrutiny only may be necessary. There is another disadvantage: the result will not in all cases agree with that of the proposed method. For, let us suppose that $a$, $b$, $c$ are all positive, and that $a$ is the least of the three, and at the same time that $2c$ is less than $a+b$. On the method proposed, as we have already seen, C would be elected, but on the method now under discussion B would be elected. For the scores of A and B at the first scrutiny are $2N-b+c$, $2N-c+a$, respectively, and the first of them is the smallest, because $2c$ is less than $a+b$, and therefore $c-b$ is less than $a-c$. Thus A would be thrown out at the first scrutiny, and a second scrutiny would be held to decide between B and C, and B would win because $a$ is positive. Thus the result is that which would follow from abandoning the proposition "A is better than B," which is affirmed by a majority of $2c$, whereas the result of the proposed method is that which would follow from abandoning the proposition

"B is better than C," which is affirmed by a majority of $2a$, which is smaller than the former majority.

There is, however, one point in favour of the modified method. The first scrutiny will at once give us the values of the three differences $b—c$, $c—a$, $a—b$. From these, of course, we cannot find $a$, $b$, $c$. In the modified method, however, a second scrutiny is always necessary, and this will at once give us the value of one of the three $a$, $b$, $c$. Having already found the three differences, we can at once find each of the quantities $a$, $b$, $c$, and hence we can ascertain if the result is demonstrably correct. Thus if the modified method be used, we can always ascertain, by a simple calculation, whether the result is perfectly satisfactory or not. The same remark applies to the proposed method in those cases in which two scrutinies are necessary.

## Algebraic Analysis.

Before leaving the case in which there are three candidates only, it may be of interest to give a short algebraical analysis of the question. As before, let the compound symbol AB stand for the number of electors who put A first and B second, and similarly for other cases. Let us suppose, as is clearly possible, that six quantities, $a, b, c, \alpha, \beta, \gamma$, are found from the following equations:

$$\begin{array}{lll} AB=\beta+c & BC=\gamma+a & CA=\alpha+b \\ AC=\gamma-b & BA=\alpha-c & CB=\beta-a \end{array}$$

Also let us suppose that 2N denotes the whole number of electors, which is clearly equal to $2(\alpha+\beta+\gamma)$, then the states of the poll on the different modes of election which have been discussed are as shown in the following table:—

| Analysis of Votes. | Single Vote. | Double. | Borda. | Condorcet. | | |
|---|---|---|---|---|---|---|
| A $\left\{\begin{array}{l} AB=\beta+c \\ AC=\gamma-b \end{array}\right\}$ | $\beta+\gamma-b+c$ | $N+\alpha$ | $2N-b+c$ | * | $N-b$ | $N+c$ |
| B $\left\{\begin{array}{l} BC=\gamma+a \\ BA=\alpha-c \end{array}\right\}$ | $\gamma+\alpha-c+a$ | $N+\beta$ | $2N-c+a$ | $N+a$ | * | $N-c$ |
| C $\left\{\begin{array}{l} CA=\alpha+b \\ CB=\beta-a \end{array}\right\}$ | $\alpha+\beta-a+b$ | $N+\gamma$ | $2N-a+b$ | $N-a$ | $N+b$ | * |
| $2N=2(\alpha+\beta+\gamma)$ | 2N | 4N | 6N | 2N | 2N | 2N |

In the first column is set out an analysis of the votes. In the second is the result of the poll on the single vote method. For instance, in the first line we have the quantity $\beta+\gamma-b+c$, which is the sum of AB and AC, *i.e.*, it denotes the number of electors who put A first. In the third column is the result of the poll on the double vote system, in which each elector has two votes. For instance, in the first line we have $N+\alpha$, or what is the same, $2\alpha+\beta+\gamma$, and this is equal to AB+AC+BA+CA, *i.e.*, it denotes the number of electors who put A first or second. In the fourth column is the result of the poll on Borda's method. For instance, in the first line we have $2N-b+c$, and this is equal to 2AB+2AC+BA+CA, as it ought to be. It is also seen at once that $2N-b+c$ is the sum of the two numbers in the first line in the second and third columns. This shows the truth of what was stated above, viz., that the poll on Borda's method is the aggregate of the polls on the single and double vote systems. In the fifth, sixth, and seventh columns, under the heading Condorcet, are set down the states of the poll on the supposition that each of the candidates, A, B, C, is excluded in turn. Thus, if A be supposed excluded for a moment, we have $N+a$ votes for B in preference to C, and consequently $N-a$ for C in preference to B. For $N+a$ is equal to AB+BC+BA, as it ought to be. Thus it is clear that $2a$ is the majority for B as against C, so that the letters $a, b, c$, have the same meaning as in the previous part of this paper. It is clear too, as has been proved before, that the number in any row in the column headed Borda, is the sum of the two numbers in the same row in the columns headed Condorcet.

The result of the method of election proposed in this paper depends solely upon the numbers $a, b, c$. The same is true of the method of Borda. On the other hand, the result of the double vote method depends solely on the values of $\alpha, \beta, \gamma$. Consequently, whatever be the result of the proposed method or of Borda's method we can clearly construct cases in which the result of the double vote method shall be what we please. The same is true of the single vote method; for although the result of the single vote method depends upon $a, b, c$ as well as upon $\alpha, \beta, \gamma$, it is easy to see that we can choose $\alpha, \beta, \gamma$ so as to eliminate the effect of the quantities $a, b, c$, whatever may be the values of the latter. The results of the Venetian method and of Ware's method depend on the values of $a, b, c$ as well as upon those of $\alpha$, $\beta$, $\gamma$, so that although for given values of $a$, $b$, $c$ we cannot bring about any result we please, still we can choose $\alpha$, $\beta$, $\gamma$ so as to bring about a result different from the true one. This, of course, is to be done by choosing $\alpha$, $\beta$, $\gamma$, so that the best candidate is thrown out at the first scrutiny. We have already seen that this is possible.

It is clear that no one of the quantities $\beta + \gamma$, $\gamma + \alpha$, $\alpha + \beta$ can be negative. For we have $\beta + \gamma =$ BC + CB, and BC, CB can neither of them be negative. Again, $\beta + \gamma =$ N — $\alpha$; thus $\alpha$ cannot be greater than N. So also $\beta$, $\gamma$ can neither of them exceed N. Since $\beta + \gamma$ cannot be negative, $\beta$ and $\gamma$ cannot both be negative; thus one only of the three $\alpha$, $\beta$, $\gamma$ can be negative. If $\alpha$ be negative it is clear that the numerical value cannot exceed N, for $\alpha + \beta$ cannot be negative, and $\beta$ cannot exceed N. So for $\beta$ and $\gamma$. Thus no one of the three $\alpha$, $\beta$, $\gamma$ can numerically exceed N, and one at most can be negative.

The limits between which $a$, $b$, $c$ must lie are at once found from the consideration that AB, AC, &c., must none of them be negative. Thus $a + \gamma$, $\beta - a$ can neither of them be negative; thus $a$ cannot be less than $-\gamma$ nor greater than $\beta$. Hence, *a fortiori*, no one of the three $a$, $b$, $c$, can be numerically greater than N. This last result is obvious from the fact that no one of the numbers in the columns headed "Condorcet" can be negative.

Formal demonstrations will now be given of a few results.

(i.) If any candidate have less than N votes on the double vote method, he ought not to be elected.

This has already been seen, but the following proof is given. Suppose A has less than N votes; then $\alpha$ must be negative, and therefore $c$ must be negative and $b$ positive. Thus A is worse than B, and also worse than C.

(ii.) Even if every elector put A in the first or second place it does not follow that A ought to be elected.

For if A has no third places we must have BC = O and CB = O, thus $a = \beta = -\gamma$. Suppose $\beta$ positive and therefore $\gamma$ negative. Then by preceding case C ought to go out and A or B ought to win as $c$ is positive or negative. Now $c$ may be negative so that B may win; for the only conditions with reference to $c$ are that $c$ must be greater than $-\beta$ and less than $\alpha$, and as $\beta$ is positive it is clear that $c$ may be negative.

(iii.) It is impossible to arrive at the true result by merely counting the number of first places, the number of second places, and the number of third places for each candidate.

This result seems obvious enough after what has been given. It may, however, be formally proved as follows.

Let $A_1$, $A_2$, $A_3$ denote the numbers of first, second, and third places respectively for A, and let corresponding meanings be given to $B_1$, &c., $C_1$, &c. Then we have

$$A_1 = \beta + \gamma - b + c$$
$$A_2 = 2a + b - c$$
$$A_3 = \beta + \gamma$$

with corresponding equations for B's and C's. We see at once from these equations that it is impossible to find $a, b, c$ even if $A_1$, $A_2$, $A_3$, $B_1$, &c., be all given. We can, however, find $\alpha, \beta, \gamma$ and the three differences $b - c$, $c - a$, $a - b$, viz., the results are

$$\alpha = N - A_3,\ \beta = N - B_3,\ \gamma = N - C_3$$
$$b - c = A_3 - A_1,\ c - a = B_3 - B_1,\ a - b = C_3 - C_1,$$

where $2N = A_1 + B_1 + C_1 = A_3 + B_3 + C_3 \quad . \quad . \quad . \quad . \quad$ (i)

thus any five of the quantities $A_1$, $B_1$, $C_1$, $A_3$, $B_3$, $C_3$, may be chosen at pleasure; the sixth and N are then determined by the conditions (i) and $A_2$, $B_2$, $C_2$ are then given by the equations

$$A_2 = 2N - A_1 - A_3, \text{ \&c.}$$

(iv.) If there be a demonstrably correct result, say A better than B and B better than C, so that $c$, $a$, are positive and $b$ negative, then if Ware's method be wrong, Venetian method is right, and if Venetian method be wrong, Ware's method is right.

For if Ware be wrong A must be lowest on the single vote method, and therefore we must have

$$\alpha + \beta - a + b > \beta + \gamma - b + c$$
$$\text{or } \alpha > \gamma + a + c - 2b$$

*i.e., a fortiori* $\alpha > \gamma$ because $a$, $c$ are positive and $b$ negative. Thus A cannot be lowest on double vote method, so that A will win on the Venetian method. Again, if Venetian be wrong, A must be lowest on double vote method, and therefore we must have $\gamma > \alpha$ and therefore $\beta + \gamma - b + c > \alpha + \beta - a + b$ because $a, c$ are positive and $b$ negative. Thus A cannot be lowest on single vote method, so that A will win on Ware's method.

(v.) If we agree to accept the proposed method as correct in all cases, then the conclusions of the last proposition will be true in all cases.

For, in the demonstration of the last proposition, the essential condition is that $a + c - 2b$ should be positive. Now, if we suppose as before that the accepted result is A better than B, and B better than C, we must have $a, b, c$ all positive and $b$ the smallest of the three, so that it is clear that $a + c - 2b$ is positive.

Comparing then Ware's method with the Venetian method, we see that both may be right, or one wrong and one right, but both cannot be wrong; so that, if these two methods agree, the result cannot be shown to be wrong. If, however, they do not agree, we cannot tell which is right without in effect having recourse to the proposed method.

(vi.) If $a = b = c$, single and double vote methods give different results.

For A's scores on the two methods will be respectively $N - \alpha$ and $N + \alpha$. Thus, if $\gamma > \beta > \alpha$, the candidates are in the order A, B, C on the single vote method, and in the order C, B, A on the double vote method. In this case Borda's method leads to a tie, and consequently the proposed method also. Ware elects A or B as $c$ is positive or negative, and Venetian method elects C or B as $a$ is negative or positive. Thus, in this case, Ware and Venetian method give different results.

(vii.) If $\alpha = \beta = \gamma$, double vote method, and therefore also Venetian method, gives a tie; single vote method and Borda lead to same result; but Ware and proposed method will not necessarily lead to same result. If one only of the three, $b - c$, $c - a$, $a - b$, be negative, Ware and proposed method will lead to same result; but if two be negative the results may or may not agree.

(viii.) If AB = AC, BC = BA, CA = CB, all the methods will give the same result, and that result will be demonstrably correct.

This is the case in which the strong supporters of each candidate are equally divided as to the merits of the remaining candidates. In this case we have

$$a = \beta - \gamma, b = \gamma - \alpha, c = \alpha - \beta,$$

and A's scores on the single, double, and Borda's method are respectively $2\alpha$, $N + \alpha$, $N + 3\alpha$. Thus, if $\alpha > \beta > \gamma$, it is obvious that each of these methods will put A first, B second, and C third, and it is clear that this result is correct, for $a, c$ are positive and $b$ negative. It is at once seen that all the methods which have been discussed will lead to the same result in this case.

(ix.) If we suppose that

$$\alpha = \frac{N}{3} + p\,(b - c),\ \beta = \frac{N}{3} + p\,(c - a),\ \gamma = \frac{N}{3} + p\,(a - b),$$

then A's scores on the single, double, and Borda methods will be respectively

$$\frac{2N}{3} - (p + 1)\,(b - c),\ \frac{4N}{3} + p\,(b - c),\ 2N - (b - c).$$

Hence we see that

If $p < o$ and $> -1$, the results of all three methods will be the same.

If $p < -1$, double and Borda methods will give the same result, which will be opposite to that of single method.

If $p > o$, single and Borda methods will give the same result, which will be opposite to that of double method.

Thus, if $p > o$ or $< -1$, single and double methods will give different results. If we suppose that $b$, $c$ are positive and $a$ negative, and also that $2b < c + a$, then it may be shown that these different results will both be wrong.

## Cases of More than Three Candidates.

It remains now to state and examine the method proposed for the case in which there are more than three candidates.

A series of scrutinies are held on Borda's system of voting, and all candidates who on any scrutiny have not more than the average number of votes polled on that scrutiny are excluded. As many scrutinies are held as may be necessary to exclude all but one of the candidates, and the candidate who remains uneliminated is elected.

The method proposed cannot lead to the rejection of any candidate who is in the opinion of a majority of the electors better than each of the other candidates, nor can it lead to the election of a candidate who is in the opinion of a majority worse than each of the other candidates. These results are an extension of those already proved for the case of three candidates, and they may be proved as follows:— As before, let 2N be the number of electors, and let the candidates be denoted by A, B, C, D, &c. Let the compound symbol $ab$ denote the number of electors who consider A better than B, and let corresponding meanings be given to $ac$, $ad$, $ba$, &c., so that $ba$ will denote the number of electors who prefer B to A, and we shall, therefore, have $ab + ba$

$= 2N$. Now suppose that at the commencement of any scrutiny the unexcluded candidates are A, B, C, . . . . P, then the score of A on that scrutiny will be

$$ab + ac + ad + \; . \; . \; . \; . + ap.$$

For suppose that there are $n$ unexcluded candidates, and consider a voting paper on which A now occupies the $r$th place. For this A gets $n - r$ votes. Now on this paper A stands before $n - r$ other candidates. Thus the $n - r$ votes which A receives may be considered each as due to the fact that A stands before one of the following $n - r$ candidates. Thus we see that on any one voting paper A receives one vote for every candidate placed after him. Summing up for all the voting papers, we see that A receives one vote for each candidate placed after him on each paper. Now $ab$ denotes the number of times B is placed after A on all the papers, and similarly for $ac$, $ad$, &c. Thus it is clear that A's score is

$$ab + ac + ad + \; . \; . \; . \; . + ap.$$

This result was stated by Borda,* but proved only for the case of three candidates.

The whole number of votes polled is

$$2N\,(1 + 2 + 3 + 4 \ldots + n{-}1)$$

or $Nn\,(n{-}1)$. Thus the average polled by all the candidates is $N(n{-}1)$. Now let us suppose that there is a majority for A as against each of the other candidates, then each of the $n{-}1$ numbers $ab$, $ac$, $ad$, . . . . $ap$ is greater than N; thus the sum of these numbers, which is equal to A's score, is necessarily greater than $(n{-}1)$ N, that is, greater than the average score. Thus A will be above the average on every scrutiny, so that he must win on the proposed method.

Next, let us suppose that there is a majority for each of the other candidates against A. Then each of the numbers $ab$, $ac$, . . . $ap$ is less than N, and therefore their sum, which is equal to A's score, is less than $(n{-}1)$ N, that is, less than the average score. Thus A is below the average, and will, therefore, be excluded at the first scrutiny.

The results which have just been proved are particular cases of a more general theorem, which may be enunciated as follows :—

If the candidates can be divided into two groups, such that each candidate in the first group is, in the opinion of a

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* *Mémoires de l'Académie Royal des Sciences*, 1781, p. 663.

majority of the electors, better than each of the candidates in the second group, then the proposed method cannot lead to the election of a candidate of the second group.

The results which have just been proved are obtained from the above by supposing, first, that the first group contains one candidate, and the second group all the rest; and second, that the first group contains all but one of the candidates, and the second group the remaining candidate.

Let the first group consist of the $l$ candidates, A, B, C, &c., and let the second group consist of the $m$ candidates, P, Q, R, &c., and let $l + m = n$, so that $n$ is the whole number of candidates. Because each of the candidates A, B, C, &c., is better than each of the candidates P, Q, R, &c., each of the numbers $ap$, $aq$, $ar$, &c. . . . $bp$, $bq$, &c. . . . &c., is greater than N. Now the scores of A, B, C, D, &c., at the first scrutiny are respectively

$$\begin{array}{llllllllll}
* & ab & + ac & + ad & + \&c. \ldots\ldots & + ap & + aq & + ar & + \&c. \\
ba & * & + bc & + bd & + \&c. \ldots\ldots & + bp & + bq & + br & + \&c. \\
ca & + cb & * & + cd & + \&c. \ldots\ldots & + cp & + cq & + cr & + \&c. \\
da & + db & + dc & * & + \&c. \ldots\ldots & + dp & + dq & + dr & + \&c. \\
 & \&c. & & \&c. & & & \&c. & \&c. &
\end{array}$$

If we add together all these numbers, we shall get the sum of the scores of A, B, C, D, &c. Now the numbers in the first $l$ columns can be arranged in pairs, such as $ab$, $ba$, and $ab + ba = 2\mathrm{N}$, and then are $\frac{1}{2}\, l\, (l - 1)$, of these pairs; thus, the sum of the first $l$ columns is $\mathrm{N}l(l - 1)$. Again, the numbers in the last $m$ columns are each greater than N, and there are $lm$ of these numbers; thus, the sum of the last $m$ columns is greater than $\mathrm{N}lm$. Thus, the sum of all the numbers is greater than $\mathrm{N}l(l - 1) + \mathrm{N}lm$; that is, than $\mathrm{N}l(l + m - 1)$; that is, greater than $\mathrm{N}l(n - 1)$. Thus the sum of the scores of the $l$ candidates of the first group is greater than $\mathrm{N}l(n - 1)$. Hence the average score of the candidates of the first group is greater than $\mathrm{N}(n - 1)$. Hence the candidates of the first group cannot all be rejected at the first scrutiny. By the same reasoning it follows that those of the first group who survive cannot all be rejected at the second scrutiny; and so on. Thus some candidate of the first group must win on the proposed method; or, in other words, no candidate of the second group can be elected.

If the candidates can be divided into two groups in the manner just indicated, it is quite clear that no candidate of the second group ought to win. At the same time, whichever of the candidates of the first group wins, the result cannot be shown to be erroneous. If the division into groups can be made in more than one way it is clear that the last statement applies only to the smallest group of the first kind. Now in the proposed method the successful candidate must belong to the smallest group of the first kind. Hence then it is clear that the result of the proposed method cannot be shown to be erroneous in any case.

It is clear that no candidate can have more than N $(2n - 2)$ votes on any scrutiny, $n$ being as before the number of unexcluded candidates at the commencement of that scrutiny. For a candidate could only have this number by obtaining the first place on each voting paper.

Again, if any candidate obtain N $(2n - 3)$ votes on any scrutiny, there is an absolute majority in his favour, so that we can at once elect him. For if a candidate were not put first on half the papers, he could not have so many as $(n - 1)$ N + $(n - 2)$ N votes, this being the number he would have if he were put first on one half of the papers and second on the other half. It is clear, too, that if any candidate has less than N votes there is an absolute majority against him; for if a candidate has less than N votes, he must be last on at least half of the papers. These results are not of much use except in the case of three candidates; for if there be more than three candidates, it is only in cases of remarkable unanimity that a candidate can have so many as N $(2n - 3)$, or so few as N votes. If, however, there be three candidates only, the above results may be stated as follows:—The average is 2N; the largest number of votes any one candidate can have is 4N; if any candidate has 3N votes, or more, there is an absolute majority for him, and we can elect him at once, no matter whether the second candidate is above the average or not; if any candidate has less than N votes, there is an absolute majority against him, so that the result of the proposed method is demonstrably correct.

In the case of any number of candidates it will sometimes save a great deal of trouble if we first examine if there be an absolute majority for or against any candidate. This is easily done, and the results arrived at in the inquiry will be of use in carrying out the proposed method, if such be found necessary. For let $A_1$, $A_2$ . . . $A_n$ denote the numbers of papers on which A occupies the first, the second . . . the last or $n$th place, and let similar meanings be given to $B_1$, $B_2$, &c., $C_1$, &c. If $A_1$ be greater than N, there is an absolute majority for A, and we may at once elect him. If $A_n$ be greater than N, there is an absolute majority against A, and we may at once exclude him. If neither of these results hold good for any candidate, we must use the proposed method in its general form. Now A's score on that method is

$$(n - 1)A_1 + (n - 2)A_2 + \dots + (n - r)A_r + \dots + A_{n-1}.$$

Thus to find A's score we must find $A_2$, $A_3 \dots A_{n-1}$. Now to find these it is not necessary to count all the votes for A. For we have

$$A_1 + A_2 + A_3 + \dots + A_n = 2N,$$

and $A_1$, $A_n$ having been already found, we see that it is sufficient to calculate any $n - 3$ of the $n - 2$ quantities, $A_2$, $A_3 \dots A_{n-1}$, and the remaining one can then be found from the above equation.

It would, however, in practice be better to calculate each of the $n$ quantities, $A_1$, $A_2 \dots A_n$, and then to use the above equation as a test of the accuracy of the counting of the votes. Similar remarks apply to the numbers $B_1$, $B_2 \dots B_n$, $C_1$, $C_2 \dots C_n$, &c.

We have also $n$ equations of the former

$$A_r + B_r + C_r + \dots = 2N$$

where $r$ may have any one of the values 1, 2, 3 . . . $n$. This gives us $n$ independent tests of the accuracy of the enumeration of the votes. In fact, if we arrange the $n^2$ quantities, $A_1$, $A_2 \dots A_n$, $B_1$, &c., in the form of a square array

$$\begin{array}{llll} A_1, & A_2, & A_3, & \&c. \\ B_1, & B_2, & B_3, & \&c. \\ C_1, & C_2, & C_3, & \&c. \\ \&c., & \&c., & \&c. & \end{array}$$

the sum of every row and of every column ought to be 2N, so that we have altogether $2n - 1$ independent tests of the accuracy of the enumeration of the votes.

The proposed method is not so laborious as might appear at first sight. The number of scrutinies will not usually be large; for we may reasonably expect to halve the number of candidates at each scrutiny. At each scrutiny we reject all who are not above the average. Now in the long run we may expect to find as many below as above the average on a poll. Thus, if there be eight candidates we should not, on the average, require more than three scrutinies. There can be no doubt, however, that the method would be tedious if the number of electors were very large, unless the number of candidates was very small indeed. In cases where the number of electors is large Ware's method has great practical advantages; for in that method we only require to count one vote for each paper examined at each scrutiny, and at every scrutiny except the first the number of papers to be examined is but a small fraction of the whole number of papers.

### Condorcet's Theoretical Method.

A method of election was described by Condorcet in 1785, but on account of its complexity it was never proposed for actual use. On this account, and in order to distinguish it from Condorcet's practical method (which has been already described), I propose to call it Condorcet's theoretical method. This method is described by its author in the following terms:—

"There exists but one rigorous method of ascertaining the wish of the majority in an election. It consists in taking a vote on the respective merits of all the candidates compared two and two. This can be deduced from the lists upon which each elector has written their names in order of merit."

"But, in the first place, this method is very long. If there are only twenty candidates, in order to compare them two and two we must examine the votes given upon one hundred and ninety propositions, and upon seven hundred and eighty propositions if there are forty candidates. Often, indeed, the result will not be as satisfactory as we could wish, for it may happen that no candidate may be declared by the majority to be superior to all the others; and then we are obliged to prefer the one who is alone judged superior to a larger number; and amongst those who are judged superior to an equal number of candidates, the one who is either judged superior by a greater majority or inferior by a smaller. But cases present themselves where this preference is difficult to determine. The general rules are complicated and embarrassing in application." (*Œuvres de Condorcet*, vol. xv., pp. 28, 29.)

By this method Condorcet showed that the single vote method and the methods of Ware and Borda are erroneous. I do not think however, that any one has hitherto noticed that Borda's method may lead to the rejection of a candidate who has an absolute majority of the electors in his favour as against all comers. It has also been shown above by the help of this theoretical method that Condorcet's practical method is erroneous. Thus it will be seen that the theoretical method is of use in testing the accuracy of other methods. From the description which has been given above, however, it is not clear what the result of the theoretical method is, even in the simplest cases, when discordant propositions are affirmed, for if there be three candidates only, and with the notation already used, we have $a = 1$, $b = 2$, $c = 3$, each candidate is superior to one other candidate, and A is superior by most, whilst C is inferior by least. Thus, according to the above description, it is not certain which of the two, A or C, wins. In another passage, however,* Condorcet explains how he deals with any case of three candidates, and the process he adopts in the case of inconsistent propositions is to reject the one affirmed by the smallest majority. This is exactly the process which has been described above, and which was shown to be in accordance with the method proposed. Thus it is clear that in the case of three candidates the result of the proposed method will always be the same as that of Condorcet's theoretical method.

The general rules for the case of any number of candidates as given by Condorcet† are stated so briefly as to be hardly intelligible. Moreover, it is not easy to reconcile these rules with the statements made in the passage quoted above, and as no examples are given it is quite hopeless to find out what Condorcet meant.

## Comparison of Proposed Method with Condorcet's Theoretical Method.

Comparing the method proposed in this paper with Condorcet's theoretical method, we see that, so far as any conclusion can be drawn from the votes of the electors the two methods always agree. In those cases in which no conclusion can be drawn from the votes the results of the two methods will not always be the same. It is equally impossible to prove either of these results wrong. Con-

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* *Œuvres*, vol. xiii., p. 259.

† *Essai sur l'application de l'analyse à la probabilité des decisions rendues à la pluralité des voix*, pp. 125, 126.

dorcet's method always shows whether the result is incapable of being proved wrong or not, but the proposed method gives us no information on this point. With the proposed method, however, there is no difficulty in arriving at the result in any case, whereas Condorcet's method is, by his own admission, so complicated as to be quite impracticable. Condorcet returns the candidate who is superior to the largest number of other candidates, without reference either to the numbers of votes by which the candidate is superior to those other candidates, or to the number of votes by which the candidate is inferior to the remaining candidates. Now in the proposed method both these elements are taken into consideration. Each candidate is, in fact, credited with the numbers of votes by which he beats all candidates he is superior to, and is debited with the numbers of votes by which he is beaten by all candidates he is inferior to. All candidates who have the balance against them are excluded, and the election then proceeds as if the remaining candidates were the only ones eligible.

It seems clear, then, that the proposed method is quite as rigorous as that of Condorcet. It gives the same result as Condorcet's in the case of three candidates, and it agrees therewith in all cases so far as any conclusion can be drawn from the votes. In those cases in which no valid conclusion can be drawn from the votes the two methods may not agree, and although nothing can be proved one way or another in these cases, the principles on which the proposed method is founded seem quite as sound as those of Condorcet's method. The proposed method has, however, great practical advantages over Condorcet's method, for the process of arriving at the result is the same in all cases; the operations throughout are of the same kind. The number of numerical results which have to be arrived at is much smaller than in Condorcet's method. For instance, if there be sixteen candidates we should expect, in the long run, to have four scrutinies, involving thirty numerical results, whereas Condorcet's method would require the computation of the votes for and against one hundred and twenty different propositions. When the numerical results are arrived at there is not the slightest difficulty in applying them, whereas in Condorcet's method the rules are very complicated. It may be claimed, then, that the proposed method has all the rigour of Condorcet's method and none of its practical difficulties.

## Incomplete Voting Papers.

There is a point of some practical importance to be considered in connection with the proposed method. If the number of candidates was large, some of the electors might not be able to make out a complete list of the candidates in order of preference. We have then to consider how voting papers, on which the names are not all marked in order of preference, are to be dealt with. Such a voting paper may be called incomplete. In order to examine this question, let us first suppose, for the sake of simplicity, that there are only three candidates A, B, C, and that the votes tendered are of one of the forms AB, BA, C, that is to say, that all the electors who put A first put B second, that all who put B first put A second, and that all who vote for C mark no second name. In accordance with the proposed method, for each paper of the form AB, two votes would be given to A and one to B; and for each paper of the form BA, two votes would be given to B and one to A. The question arises, however: is a paper of the form C, that is, a plumper for C, to be counted as one vote or as two votes for C? If it be counted as one vote only, it is clear that C might be defeated even if he had an absolute majority of first votes in his favour. For if we suppose $AB = BA = a$, and $C = c$, it is clear that the scores of A and B will each be equal to $3a$, and that of C to $c$. Thus C will be defeated unless $c > 3a$; but if $c > 2a$, there is an absolute majority for C. Hence, then, we may be led into error if each plumper for C be counted as one vote only. If, on the other hand, a plumper be counted as two votes, it is clear that C might win even if there were an absolute majority against him. For the score of C will now be $2c$, and C will win if $2c > 3a$. But if $2c < 4a$, there is an absolute majority against C. Thus we should also be led into error if each plumper be counted as two votes. If, however, we agree to count a plumper as three halves of a vote, neither of these errors could occur. This course is readily seen to be the proper one in any case of three candidates, for it clearly amounts to assuming that the electors who plump for C are equally divided as to the merits of A and B. For if $a^1$, $b^1$, $c^1$ denote the numbers of plumpers for A, B, C respectively, and if we agree to consider all the electors who plump for A as being equally divided as to the merits of B and C, the effect of the $a^1$ plumpers for A would be to give $2\,a^1$ votes to A, and $\frac{1}{2}\,a^1$ each to B and C. Now, as we are only concerned with the differences of the totals polled for each candidate, we see that the result of the first scrutiny will be the same if we take away $\frac{1}{2}$ $a^1$ votes from each candidate. Thus the result will come out the same if we give $\frac{3}{2}$ $a^1$ votes to A, and none to B or C, so far as the plumpers are concerned. Similarly the result will not be altered if the $b^1$ plumpers for B be counted, as $\frac{3}{2}$ $b^1$ votes for B and nothing for C and A, and so for C's plumpers. Thus the final result will be in accordance with the views of the electors, if each plumper be reckoned as three halves of a vote

The assumption that the electors who plump for A are equally divided as to the merits of B and C, appears to be perfectly legitimate, for the electors have an opportunity of stating their preference, if they have one, and as they have, in the case supposed, declined to express any, it may be fairly concluded that they have none.

At the final scrutiny (if held), all plumpers for the candidate who has been rejected will have no effect.

If there be more than three candidates, and incomplete papers are presented, we should have to make a similar assumption, viz., that in all cases where the preference is not fully expressed, the elector has no preference as regards the candidates whom he has omitted to mark on his voting paper. Thus, for example, if there be four candidates, A, B, C, D, a plumper for A ought to count as two votes for A and none for B, C, D. Again, a voting paper on which A is marked first and B second, and on which no other names are marked, ought to count as two and ahalf votes for A and three halves of a vote for B. If there be more than four candidates the varieties of incomplete papers would be more numerous, and the weights to be allotted to each would be given by more complicated rules. Practically it would be best to count one vote for each plumper in the case in which only one candidate is marked on a voting paper; one for the last, and two for the first, when two names only are marked on a voting paper; one for the last, two for the next, and three for the first, when three names only are marked on a voting paper, and so on, giving in all cases one vote to the candidate marked lowest on any paper, and as many votes to the candidate marked first as there are names marked on the paper. By this means the rules for computing the votes would be the same in all cases and at all scrutinies. We have seen, it is true, that this method may lead to error. The error has the effect of decreasing the votes for the candidates who are marked on any incomplete paper, and it arises solely in consequence of the papers being incomplete. Thus, if the electors do not fully express their preference, the effect is to injure the chances of their favourite candidates. If, then, we adopt the plan just described for incomplete papers, it will be sufficiently simple for practical purposes, and its use will tend to elicit from electors a full statement of their various preferences.

## Cases of Equality.

No case of equality can occur in the proposed method except when all the candidates poll exactly the same number of votes on a scrutiny, for if less than the whole number of candidates have the same number of votes in any scrutiny, if that common number be not greater than the average, all the equal candidates are excluded. If it be greater, no one of them is excluded; and in either case we pass on to another scrutiny.

If on any scrutiny all the candidates poll exactly the same number of votes, that number, of course, must be the average, and it is necessary that some one should have a casting vote. If it is thought proper to do so, one casting vote can then be made to settle the election, by allowing the casting vote to decide who is to win. But if it is thought that this is giving too much weight to the casting vote, then we may permit the casting vote to decide who is to be excluded, and then proceed to a fresh scrutiny between the remaining candidates. It will be observed, however, that the chance of a casting vote being required at any scrutiny except the last, when only two candidates remain, is very minute, seeing that it depends upon all the candidates polling exactly the same number of votes on a scrutiny.

## Statement of Method.

It is convenient to give here a formal statement of the method which it is proposed should be used when incomplete papers are presented.

Each elector is furnished with a list of the candidates in alphabetical order, upon which he indicates his preference amongst the candidates by placing the figure one opposite the name of the candidate of his first choice, the figure two opposite the name of the next in order of preference, the figure three opposite the next, and so on, to as many names as he pleases.

It is, of course, unnecessary to mark all the names; it is sufficient to mark all but one. In what follows, if all the names be marked, it is unnecessary to pay any attention to the name marked lowest in order of preference.

The mode of dealing with the papers is as follows:—For the lowest candidate marked on any paper count one vote, for the next lowest two votes, for the next three votes, and so on, till the highest is reached, who is to receive as many votes as there are names marked on the paper. The total number of votes for each candidate is then to be ascertained; and thence the average number polled. All candidates who have not polled above the average are then to be excluded. If more than one candidate be above the average, then another scrutiny must be held as between all such candidates.

In counting up the votes for the second, or any subsequent scrutiny, no attention must be paid to the names of any candidates who have been excluded.

As many scrutinies as may be necessary must be held, so that finally all the candidates but one are excluded, and the last remaining candidate is elected.

## Practical Details.

In order to show precisely the amount of labour which would be required to carry out the proposed method, it may be as well to state what appears to be the most convenient way of making up the result. As in the ordinary methods, it would be necessary to have a poll-book in which to keep a tally of the votes. In this book the names of the candidates should be printed from the same type as the ballot papers are printed from. Each ballot paper should be placed with the names in a line with the corresponding names in the poll book, and the numbers written opposite to the names on each ballot paper should then be copied into the successive columns of the poll-book. In this way the risk of error in transcription would be exceedingly small, and any error which was made would be at once detected on placing the ballot paper side by side with the column in which its numbers are recorded. When this is done many of the columns would contain vacant spaces. In every vacant space in each column write a number greater by unity than the largest number copied from the voting paper into that column. After doing this add up the figures in each row; then find the mean or average of the sums. Every candidate who has a sum *equal* to or *greater* than the average is to be excluded. A little consideration will show that this process will give the same result as the method described above. When the papers have once been copied into the poll-book as just described, all subsequent scrutinies that may be necessary can be conducted without handling the voting papers again.

## Cases of Bracketing.

Under the head of "Incomplete Voting Papers" we have considered a case in which an elector does not fully express his preference. There is, however, another way in which an elector may fail to fully express his preference. An elector may have no difficulty in putting a number of candidates at the bottom of his list, and yet he may have considerable difficulty in deciding as to the precise order in which to place the candidates at the top end of his list. In such a case an elector might wish to put two or more candidates equal for the first, second, or some other place on his list. This may be called a case of bracketing. It is now to be shown that this system of bracketing can be permitted without causing any difficulty in the practical working of the system. Let us suppose that an elector brackets $m_1$ candidates for the first place, $m_2$ for the second place, and so on; so that $m_1 + m_2 + m_3 + \ldots = n$, the case in which one candidate only is put in the $r^{th}$ place being provided for by supposing $m_r = 1$. Then in the poll-book already described enter the number one for each of the $m_1$ candidates in the first bracket, the number two for each of the $m^2$ candidates in the second bracket, the number three for each of the $m_3$ candidates in third bracket, and so on. Suppose, for example, that there are seven candidates, A, B, C, D, E, F, G, and that an elector wishes to bracket B, E for the first place and A, D, F for the second place, and that he does not care to say anything about C, G. Then he would mark his paper as shown in the margin. As nothing is said about C, G, we should consider them as bracketed for the third or last place. Now in order to record this vote in the poll-book it is merely necessary, as before, to copy the column of numbers on the

2A
1B
C
2D
1E
2F
G

voting paper into a column of the poll-book, taking care to write in two 3's in the two blank spaces opposite the names C, G. After copying the numbers from each ballot-paper into the poll-book and filling up all the vacant spaces, we should add up the different rows and proceed exactly as before to ascertain the result of the election. Thus it is clear that the method of dealing with the papers is exactly the same no matter how many or how few names be marked, nor how many are bracketed in the various brackets, and that there is very little risk of error in the process.

If this system of bracketing be permitted we at once get rid of the objection that the proposed method could only be used in a highly educated constituency, because it is only highly educated electors who can possibly arrange the candidates in order of merit. The method can easily be used by the most ill-informed electors. In fact, an elector, if he so pleased, could vote in exactly the same manner as in elections under the common "majority" system of voting in cases where there are several candidates—that is, the elector may simply cross out the names of all the candidates he objects to and leave uncancelled as many names as he pleases. In such a case the uncancelled names would all be considered bracketed for the first place, and the cancelled ones as bracketed for the second or last place.

Exactly as in the case of incomplete papers previously discussed, it is easy to see that the method just given is not strictly accurate, that the strictly accurate method would be too complicated for practical purposes, and that the error has the effect of decreasing the chances of success of the favourite candidates of the elector who resorts to bracketing. In fact it may be shown that the numbers which ought strictly to be entered in the poll-book for the candidates in the successive brackets are

$$0, \frac{m_1}{2} + \frac{m_2}{2}, \quad \frac{m_1}{2} + m_2 + \frac{m_3}{2}, \ldots . \qquad (1)$$

$$\frac{m_1}{2} + m_2 + m_3 + \ldots + m_{r-1} + \frac{m_r}{2}, \ \&c.$$

Now the plan just described comes to the same thing in the end as entering instead of these the numbers

$$0, \ 1, \ 2, \ \ldots . \ (r-1), \&c. \qquad (2)$$

and as no one of the numbers $m_1$, $m_2$, $m_3$, &c., can be less than unity, it is easy to see that no one of the numbers (2) can be greater than the corresponding one of the numbers (1), that when no bracketing occurs the two sets (1), (2), are the same, and that the two sets agree until the first bracket is reached. Now observe that the numbers entered in the poll-book are in reality negative votes, and we see at once that the moment an elector begins to bracket, he diminishes the influence of his own vote on the result of the election, and also decreases the chances of success of all candidates who on his own list are placed higher than the bracket. Each additional bracket will have precisely the same effects. Thus it is clear that the effect of the proposed method will be to discourage the practice of bracketing. If we do not wish to discourage this practice we must resort to the accurate method, and use the numbers (1) instead of (2). This is not very difficult to do, but as it introduces a new method for the bracketed votes, it would give considerable extra trouble to the officers who make up the poll-books. The most convenient way of stating the accurate method would be as follows:—For each first place count one negative vote, for each second place count in addition $\frac{1}{2}(m_1 + m^2)$ negative votes, for each third place count in addition to the last $\frac{1}{2}(m_2 + m_3)$ negative votes, for each fourth place count in addition to the last $\frac{1}{2}(m_3 + m_4)$ negative votes, and so on. As before remarked, the numbers for the successive places would be the natural numbers 1, 2, 3, 4, &c., until a bracket was arrived at. When brackets do occur we shall in general have to deal with half-votes, but no smaller fraction could occur.

## Another Method for Cases of Bracketing.

Another plan might also be adopted for dealing with cases of bracketing. It is as follows. For each candidate in the first place count one vote; for each candidate in the second place count $m_1 + 1$ votes; for each candidate in the third place count $m_1 + m_2 + 1$ votes; for each candidate in the fourth place count $m_1 + m_2 + m_3 + 1$ votes; and so on. The plan now under consideration comes to the same thing as counting for the successive places the numbers $0, m_1, m_1 + m_2, \ldots . m_1 + m_2 + \ldots + m_{r-1}$, &c. instead of the proper numbers (1). Thus the errors for the successive places are

$$0, \quad \frac{m_1 - m_2}{2}, \quad \frac{m_1 - m_3}{2}, \quad \ldots \ldots \frac{m_1 - m_r}{2}, \quad \text{\&c.}$$

Hence we see that

(i.) If the same number of candidates be bracketed for each place, the plan is accurate.

(ii.) If $m_1$ be greater than each of the numbers $m_2$, $m_3$, &c., that is, if more candidates are bracketed for the first place than for any other place—then the errors will be all positive, and the effect will be to give the elector more negative votes than he is entitled to, and, consequently, to increase unduly the chances of the candidates bracketed for the first place.

(iii.) If $m_1$ be less than each of the numbers $m_2$, $m_3$, &c.—that is, if fewer candidates are bracketed for the first place than for any other place—then the errors will be all negative, and the effect will be to give the elector fewer negative votes than he is entitled to, and, consequently, to decrease unduly the chances of the candidates placed at the top end of the elector's list.

(iv.) If $m_1$ be equal to the mean of the numbers $m_2$, $m_3$, &c., the elector will have just as many votes as he ought to have, but he will give more negative votes to some candidates and less to others than they ought to have.

(v.) If $m_1$ be not equal to the mean, then the elector will have more or less votes than he is entitled to, according as $m_1$ is greater or less than the mean.

The results just given apply to each scrutiny; but the numbers $m_1$, $m_2$, $m_3$, &c., will generally be altered at each scrutiny. Thus it is in general impossible to tell at the commencement of an election what will be the effect of different modes of bracketing. Sometimes the elector will get too many votes, sometimes too few. At some scrutinies the candidates at the top end of his list will get too many votes, and at others those at the lower end will get too many votes.

If there be one candidate only in each place except the last, or, in other words, if the only bracket be for the last place, we have the case of incomplete papers discussed above. In this case the plan just described, and the method adopted above, agree; and the effect is, as has already been pointed out, to give the elector too few votes; and this would be the case at each scrutiny, until all but one of the candidates in the bracket are rejected.

If, however, an elector bracket a number of candidates for the first place and arrange all the rest in order of merit, he would get more votes than he is really entitled to and this would be the case at each scrutiny until all but one of the candidates in the bracket are rejected. Electors would very soon find this out. Each elector would ask himself the question, How must I vote in order to get as much electoral power as possible; and the answer would very soon be seen to be—I must bracket all the candidates I don't object to for the first place, and I must arrange all the rest in numerical order. Thus, instead of encouraging the electors to arrange all the candidates in order of merit, this plan would lead to each elector trying all he could to defeat objectionable candidates without expressing any opinion as to the relative merits of those he does not object to.

## Rule for Forfeit.

If the method which is proposed were adopted for parliamentary elections, it is clear that the number of candidates would be very much greater than at present. In order to prevent the number becoming so great as to make the election unmanageable, it is necessary to provide some method for keeping the number of candidates within reasonable bounds. Such a provision exists for the method now in use. It is that any candidate who fails to obtain one-fifth of the number of votes polled by the lowest successful candidate forfeits the deposit which he has lodged with the returning-officer. This rule is, of course, purely empirical, and we must fix upon some rule of the same kind for the proposed method. I will first state a rule for the method as first described—*i.e.*, when positive votes are used. This rule is as follows:—

If at the first scrutiny any candidate has a number of votes which is less than half the number of votes polled by the candidate who is highest at the first scrutiny, he shall forfeit his deposit.

In the mode of applying the method which is most convenient in practice this rule takes a somewhat more complicated form, as follows:—

If at the first scrutiny any candidate has a number of votes which, together with a number which is equal to half the number of electors, exceeds half the number of votes polled by the candidate who has the smallest number of votes by the average for the first scrutiny, he shall forfeit his deposit.

## Case of Several Vacancies.

Hitherto we have supposed that there is only one vacancy to be filled. If there be more than one vacancy we have to settle a most important question before we can consider what method of election is to be adopted. This question is as follows:—Is the majority of the electors to fill the whole of the vacancies, or are the successful candidates supposed to represent the different sections of the electoral body? The first case is that of the selection by a board of governors of officers to fill various offices. No question of representation is involved, but simply the selection of those persons most fit, in the opinion of the whole electoral body, to fill the different offices. The second case is that of the selection of representatives by a large electoral body. In the first case the whole electoral body has to decide for itself once for all, and the majority must rule. In the second case the electoral body has to select representatives, who are to decide and act for it in a variety of matters; and in order that the decision may be as far as possible in accordance with the views of the electoral body, it is necessary that all the different sections thereof should, as far as possible, be represented.

In the first case there is only one method of arriving at the correct result, and the method is to fill each vacancy separately. Thus one person must be elected by the method described above; then by means of the same set of voting papers we must proceed to a second election for the next vacancy, and so on till all the vacancies are filled. After each vacancy is filled we must of course suppose the name of the successful candidate erased from all the voting papers.

The second case—that of the selection of representatives—has been considered by Hare, Andræ, and other writers. It is not proposed here to discuss this question beyond pointing out that it follows from the principles which have been established in this paper that the process of "elimination" which has been adopted by all the exponents of Hare's system is not satisfactory.